APPROXIMATION THEORETIC ASPECTS OF THE POST-WIDDE INVERSION OF LAPLACE TRANSFORM

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CERTIFICATE

Certified that the work presented in this thesis entitled 'SOME APPROXIMATION THEORETIC ASPECTS OF THE POST-WIDDER INVERSION OF LAPLACE TRANSFORM' by Sri Om Prakash Singh has been carried out under my supervision and that this has not been submitted elsewhere for a degree or diploma.

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CONTENTS

Chapter	0	Introduction and Contents of the Thesis	
	0.1	Introduction	1
	0.2	Contents of the Thesis	7
Chapter	I	Basic Results and Linear Combination	
	1.1	Introduction	10
	1.2	Qualitative Approximation Properties	11
	1.3	Some Evaluations and a Recurrence Formula for Moments	14
	1.4	A Few Basic Lemmas	16
	1.5	Basic Convergence, Error Estimates and Asymptotic Formulae	22
	1.6	Linear Combinations	26
Chapter	II	Best Asymptotic Constants in Error Estimates	
	2.1	Introduction	32
	2.2	An Estimation	33
	2.3	The Lipschitz-Nikolskiĭ Constants	35
	2.4	An Improved Error Estimate for Continuously Differentiable Functions	38
	2.5	Best Asymptotic Constant with Modulus of Continuity	39
Chapter	III	Direct, Inverse and Saturation Theorems	
	3.1	Introduction	45
	3.2	Commutativity Approach	46
	3.3	Limits of Certain Iterates of Post-Widder Operators	61
	3.4	Inverse and Saturation Theorems for Linear Combinations	72
Chapter	IV	Simultaneous Approximation	
	4.1	Introduction	81
	4.2	The Simultaneous Approximation Property	81
	4.3	Asymptotic Formulae in the Simultaneous Approximation	85
	4.4	Linear Combinations and Simultaneous Approximation	88
	4.5	Direct, Inverse and Saturation Theorems in the Simultaneous Approximation	91

Chapter V Analytic Functions and Post-Widder Operators

5.1 : Introduction 96
5.2 Characterization of Certain Classes of Analytic Functions 98
5.3 Convergence in the Complex Plane 102
5.4 : Effect of Singularities on the Region of Convergence 108
5.5 Best Possible Region of Convergence 112
5.6 : Some Further Results 113

References . 118

CHAPTER 0

INTRODUCTION AND CONTENTS OF THE THESIS

0.1 INTRODUCTION

An inversion formula for the Laplace-Stieltjes integral $\phi(x) = \int_0^\infty e^{-xu}\, d\alpha(u)$ was discovered by Post [66] for the case when $\alpha(u)$ was the integral of a continuous function f. The germ of the formula is contained in a letter from Stieltjes to Hermite dated August 29,1893 [letter 383, pp. 332-334, vol.2 of the collected correspondence, Paris 1905] . Post's formula is

$$(0.1) \qquad f(t) = \lim_{n \to \infty} \frac{(-1)^n}{n!} \phi^{(n)} \left(\frac{n}{t}\right) \left(\frac{n}{t}\right)^{n+1} .$$

The general case was first treated by Widder [89] and since then the operators $L_{n,x}$ defined by

$$(0.2) \qquad L_{n,x}[\phi(t)] = \frac{(-1)^n}{n!} \phi^{(n)} \left(\frac{n}{x}\right) \left(\frac{n}{x}\right)^{n+1}$$

are known as Post-Widder operators. For a given $n \in \mathbb{N}$ (the set of natural numbers) and $x \in \mathbb{R}^+ = (0,\infty)$, the operator $L_{n,x}$, is applicable to a function $\phi(t)$ provided $\phi(t)$ possesses a derivative of order n at $t = n/x$. This condition is fulfilled, for instance, when f(u) is bounded and locally integrable over $(0,\infty)$ and $\phi(t) = \int_0^\infty e^{-tu} f(u)\, du$. Then, also there holds the integral representation

$$(0.3) \qquad L_{n,x}[\phi(t)] = \frac{1}{n!} \left(\frac{n}{x}\right)^{n+1} \int_0^\infty e^{-nu/x}\, u^n f(u)\, du .$$

Using this representation Widder [89] proved the following result.

THEOREM I. If $f(u)$ is locally integrable over $\mathbb{R}^+$ and if the integral

$$\phi(t) = \int_0^\infty e^{-tu} f(u)\, du$$

converges for some $t \in \mathbb{R}^+$, then

$$(0.4) \qquad \lim_{n \to \infty} L_{n,x} [\phi(t)] = f(x),$$

for all $x \in \mathbb{R}^+$ in the Lebesgue set for $f(u)$.

The Lebesgue set for a function f is the set of points x_0 for which there holds

$$\int_{x_0}^{x} | f(u) - f(x_0) | \, du = o(| x-x_0 |) \qquad (x \to x_0).$$

Thus under the hypothesis of Theorem I it follows that (0.4) holds at all points $x \in \mathbb{R}^+$ where f is continuous and also that (0.4) holds for almost all $x \in \mathbb{R}^+$. Also, Widder showed that if f is of bounded variation in a neighbourhood of a point $x \in \mathbb{R}^+$, then

$$(0.5) \qquad \lim_{n \to \infty} L_{n,x} [\phi(t)] = \frac{f(x+) + f(x-)}{2}.$$

In the present thesis we shall be concerned with the behaviour of the Post-Widder operators with respect to the function $f(u)$. If we write

$$M_{n,x}\ [f(u)] = \frac{1}{\Gamma(n+1)} \left(\frac{n}{x}\right)^{n+1} \int_0^\infty e^{-\frac{nu}{x}} u^n f(u)\, du$$

$$= \frac{n^{n+1}}{\Gamma(n+1)} \int_0^\infty f(ux)\, u^n e^{-nu}\, du \qquad (x \in \mathbb{R}^+),$$

then $L_{n,x}\ [\phi(t)] = M_{n,x}[f(u)]$ whenever the former exists. Besides this $M_{n,x}[f(u)]$ may exist even though the Laplace transform of f does not exist. This is the case, for example, when we take $f(u) = u^{-1}$.

We shall study the behaviour of the operators $M_{n,x}$ for the general class B^* of functions f which are locally integrable on $\mathbb{R}^+$ and which for some constants $b, B > 0$ satisfy

$$f(t) = O(e^{Bt}),\ t \to \infty$$

and

$$f(t) = O(t^{-b}),\ t \to 0.$$

With these conditions, $M_{n,x}[f]$ for an $x \in \mathbb{R}^+$ exists for all real $n > \max\{B x,\ b\}$. Indeed, the functions f to which the above mentioned Theorem I of Widder is applicable are all contained in the class B^*. Also, it is easily seen that B^* is the largest class of functions f for which $M_{n,x}[f]$ is meaningful for n sufficiently large.

We notice that the action of $M_{n,x}$ on f(t) is that of a linear positive operator. An extensive study of such operators ensued with the works of Bohman [7] and Korovkin [37-40]

Korovkin gave simple conditions for the convergence of such operator sequences for bounded continuous functions. Many other researchers considered questions regarding the speed of above convergence and also the convergence problems for unbounded functions (see [68] for further references) Butzer [10] used certain linear combinations of the Bernstein polynomials to improve the order of approximation for differentiable functions. Leviatan and Müller [41] used the linear combinations of Gamma operators of Müller for the same purpose. Rathore [67] developed a general theory of linear combinations of linear positive operators.

For several sequences of linear positive operators L_n, the convergence $L_n^{(k)}(f) \to f^{(k)}$ as $n \to \infty$, known as the simultaneous approximation property, has also been studied. Such a study was pioneered by Lorentz [42] who established the above convergence for Bernstein polynomials. Similar results for various other operator sequences were obtained by Martini [50], Lupaş and Müller [48], Watanbe [87], Walk [86] and Rathore [71], etc. Rathore [71-72] initiated a study of asymptotic formulae in simultaneous approximation

The best constants related with the approximation of Lipschitz classes of functions by linear positive convolution operators have been studied by various authors. For an extensive bibliography of the work prior to 1970 we refer to Görlich and Stark [23]. Some recent work on non-convolution type operators has been done in [67,70,72].

Sikkema [78] and Esseen [18] obtained the best constants with the modulus of continuity for the Bernstein polynomials. Also, Schurer and Steutel [77] obtained such constants for the generalized Jackson operators

Jackson [27] initiated the study of direct theorems in approximation theory. Later, Bernstein after establishing his famous inequality, succeeded in proving the inverse theorems for polynomial approximation. For further details of this theory we refer to Natanson [58] and Timan [85] . Butzer and Scherer [12] have developed a general theory for obtaining inverse theorems for certain sequences of operators defined on a Banach space. DeVore [16] furnishes a good account of inverse theorems for operators which satisfy a Bernstein type inequality.

In 1941, Alexits [1] initiated the study of saturation of convolution operators by giving a characterization of the saturation class for Fejér operators. Later, Favard [19-20] gave a general formulation of the phenomenon of saturation for convolution operators. Zamansky [92] studied the saturation for convolutions with trigonometric polynomials. DeVore [15] also studied the saturation for linear positive convolution operators.

Saturation and inverse theorems for Bernstein polynomials have been intensively studied by Lorentz [43] , Suzuki and Watanbe [84] , Berens and Lorentz [3] , Lorentz and Schumaker [44] and others. Butzer and Berens [11] give a

systematic account of direct, inverse and saturation theorems for semigroups of operators Recently, Ditzian and May [17] proved a saturation theorem for linear combinations of Bernstein polynomials defined by Butzer Later, May [52] in his interesting paper obtained saturation and inverse theorems for linear combinations for the class of exponential-type operators. Also, the works of Harsiladge [24] , Buchwalter [9] , Sunouchi and Watari [82] , Newman and Shapiro [60] , Natanson [59] Sunouchi [80-81], Ikeno and Suzuki [26] , Micchelli [53] , Karlin and Ziegler [35] , Schnabl [73] and others have contributed richly to the growth of this area.

The notion of a variation diminishing operator was introduced by Schoenberg [74] where the first important result concerning the variation diminishing property for a finite matrix transformation was proved. Later, Motzkin [55] found necessary and sufficient conditions under which a matrix transformation becomes variation diminishing. Gantmacher and Krein [21] also studied matrix variation diminishing transformation Schoenberg [75] extended the studies on variation diminishing property to the case of a convolution transform (see Hirschman and Widder [25] for an interesting exposition). Karlin [31-32] , extended the variation diminishing property to general transformations induced by totally positive and sign-regular kernels.

Popoviciu [62-65] initiated a study of generalized convex functions. An extensive account of the theory is available in Karlin and Studden [34] and in Karlin [33]. Karlin discusses generalized convexity preserving transformations.

A behaviour of Bernstein polynomials for complex arguments was first discussed by Wright [91] and Kantorovitch [30]. Later, Bernstein [4-6] made a more detailed study concerning the above (Lorentz [42]). Gergen, Dressel and Purcell, Jr. [22] and Jakimovski and Leviatan [28] studied the same for the Szász operators. Similar studies about the convergence of operators in complex domain have also been made by Cheney and Sharma [14], Wood [90] and others.

0.2 CONTENTS OF THE THESIS

A chapter wise summary of the contents of the thesis is as follows :

CHAPTER I : This chapter is mainly devoted to some basic results on the operators $M_{n,x}$. In Section 1.2 we establish the variation diminishing, convexity preserving and starshape preserving properties of these operators. In the following two sections we obtain certain evaluations and preliminary lemmas of a later use. In Section 1.5 we establish the basic convergence and obtain asymptotic formulae and a few error estimates for certain functions belonging to B^*. In Section 1.6 we have defined linear combinations of the operators $M_{n,x}$ and have obtained results on the improved order of approximation.

CHAPTER II : In this chapter we have obtained Lipschitz-Nikolskiĭ constants and the best asymptotic constants in the error estimates In Section 2.3 we determine Lipschitz-Nikolskiĭ constants for the operators $M_{n,x}$. In Section 2.4 we present some results which improve the error estimates obtained in Chapter I and finally, in the last Section 2.5 best asymptotic constants with modulus of continuity have been found.

CHAPTER III : Direct, inverse and saturation theorems for the operators $M_{n,x}$ form the content of Chapter III Using a commutativity property of the operators $M_{n,x}$, we first obtain certain direct and inverse theorems in Section 3.2. In Section 3 3 we have discussed certain iterates of the operators $M_{n,x}$ and have proved that they converge to a semigroup of class (C_o). Utilizing this result we obtain a direct and inverse theorem for the operators $M_{n,x}$ and also characterize a certain class of convex functions via an inequality involving $M_{n,x}$. In Section 3.4 we obtain local inverse and saturation theorems for linear combinations of the operators $M_{n,x}$.

CHAPTER IV : In this chapter we discuss the simultaneous approximation property of the operators $M_{n,x}$. In Section 4.2 we establish the basic convergence in simultaneous approximation. In Section 4.3 we obtain an asymptotic formula in simultaneous approximation Linear combinations of the operators $M_{n,x}^{(k)}$ have been studied in Section 4.4 and the last section concerns with the direct, inverse and saturation theorems for these linear combinations.

<u>CHAPTER V :</u> This chapter is devoted to a study of the convergence of the operators $M_{n,x}$ in the complex plane. F... of all, in Section 5.2 we have characterized a certain cl... of analytic functions via a behaviour of the operators M_n In Section 5.3 we establish the convergence, $M_{n,z}[f] \to f(z)$ as $n \to \infty$ for certain analytic functions In Section 5.4 we discuss the effect of singularities of f on the above convergence. In Section 5.5 we have obtained the best possible region of the convergence corresponding to a singularity of f(z). Finally, in the last section, we use the characterization of Section 5.2 to obtain some further results on the convergence of $M_{n,x}$ in the complex plane.

CHAPTER I

BASIC RESULTS AND LINEAR COMBINATION

1.1 INTRODUCTION

The n-th Post-Widder operator $L_{n,x}$ for the Laplace transform $F(t)$ of a function $f(x)$ is defined by

$$(1.1) \qquad L_{n,x}[F(t)] = \frac{(-1)^n}{n!} F^{(n)}\left(\frac{n}{x}\right)\left(\frac{n}{x}\right)^{n+1},$$

where $x \in \mathbb{R}^+$ and n is a positive integer. If $f(x)$ is integrable on every interval (o, A), $A > o$ and $f(x) = O(e^{Bx})$, $x \to \infty$, the Laplace transform $F(t)$ of $f(x)$ exists for all $t \in (B, \infty)$ and defines an analytic function Hence $L_{n,x}[F(t)]$ is well defined for all $n > [Bx]$, the integral part of Bx. An explicitly f-dependent integral representation of $L_{n,x}[F(t)]$ obtained by Widder [89] is as follows

$$(1.2) \qquad L_{n,x}[F(t)] = \frac{1}{n!}\left(\frac{n}{x}\right)^{n+1} \int_0^\infty e^{-nu/x} u^n f(u)\, du$$

In order to make the dependence of $L_{n,x}[F(t)]$ on f more clear we write

$$(1.3) \qquad \frac{n^{n+1}}{\Gamma(n+1)x^{n+1}} \int_0^\infty e^{-nu/x} u^n f(u)du = M_{n,x}[f(t)],$$

without explicitly bringing the Laplace transform $F(t)$ of $f(x)$ into the picture.

Let B^* denote the class of all functions f defined and locally integrable on $\mathbb{R}^+$ such that for some positive constants b, B there hold

(1.4) $$f(u) = O(u^{-b}), \quad u \to o^+$$

and

(1.5) $$f(u) = O(e^{Bu}), \quad u \to \infty .$$

Then, for a $f \in B^*$ the integral defining $M_{n,x}[f]$ $(x>o)$ exists for all positive numbers $n > \max\{Bx, b\}$ At this point we also notice that if

$$f_\delta(x) = \begin{cases} o, & x < \delta \\ f(x), & x \geq \delta \end{cases} \quad , \ \delta > o,$$

where $f \in B^*$, then if $M_{n,x}[f]$ exists for some positive integer n and $x > o$, we have

$$M_{n,x}[f] = \lim_{\delta \to o} L_{n,x}[F_\delta(t)]$$

where $F_\delta(t)$ denotes the Laplace transform of $f_\delta(x)$.

In this chapter we study some basic qualitative and quantitative approximation properties of the operators $M_{n,x}$ and certain of their linear combinations.

1.2 QUALITATIVE APPROXIMATION PROPERTIES

For a real sequence $\{a_n\}$, $v\{a_n\}$, the variation of $\{a_n\}$, is defined to be the number of sign changes of the elements a_n considered in the sequential order. Thus, for example $v\{1,-2,2,o,-1,-2\} = 3$. For a function f defined on a subset of real line, $v(f)$, the variation of f, is defined by $v(f) = \sup v\{f(x_n)\}$, where the supremum is taken over all increasing sequences $\{x_n\}$ contained in the domain of f. An operator L is said to be <u>variation diminishing</u> if for all f

in the domain of L there holds $v(L(f)) \leq v(f)$, where these variations are counted on the domains of $L(f)$ and f, respectively

<u>THEOREM 1.2.1</u> The operator $M_{n,x}$ is variation diminishing.

<u>PROOF</u> In order to prove the theorem it is sufficient to show that the kernel of the operator $M_{n,x}$ is strictly totally positive (see Karlin [34]). Thus, for arbitrary increasing sequences $\{x_j\}_{j=1}^{k}$ and $\{t_j\}_{j=1}^{k}$ of positive real numbers, we have to show that the determinants

$$\Delta_k = \det\,(K_n(x_i, t_j))_{i,j=1}^{k} > 0 \qquad (k = 1,2,3,\ldots),$$

where $\quad K_n(x,t) = \dfrac{n^n}{\Gamma(n)} x^{-(n+1)}\, t^n\, e^{-nt/x}\,.$

Using the fact that the kernel $K(x,t) = e^{-t/x}$ is strictly totally positive [34, page 15], we have

$$\det\,(e^{-nt_j} / x_i)_{i,j=1}^{k} > 0,$$

where x_i's and t_j's are as above. Hence, also

$$\Delta_k = \left(\frac{n^n}{\Gamma(n)}\right)^k \left\{\prod_{j=1}^{k} t_j^n\right\}\left\{\prod_{i=1}^{k} x_i^n\right\} \det\,(e^{-nt_j} / x_i)_{i,j=1}^{k} > 0,$$

which completes the proof of the theorem

A function f is said to be k-convex if $f[x_0, x_1, \ldots, x_k] \geq 0$, where $f[x_0, x_1, \ldots, x_m]$, the m-th order divided difference is inductively defined as follows $f[x] = f(x)$, $f[x,y] = (f[x] - f[y])/(x-y)$ and

$$f[x_0, x_1, \ldots, x_m, x_{m+1}] = \frac{f[x_1, \ldots, x_{m+1}] - f[x_0, \ldots, x_m]}{x_{m+1} - x_0}, (m > 0)$$

An operator L is said to be <u>k-convexity preserving</u> if $L[f]$ is k-convex whenever f is so.

THEOREM 1.2.2 The operator $M_{n,x}$ preserves generalized convexity of any order.

PROOF Let $f \in B^*$ and be k-convex. Then $f[x_0, \ \ ,x_k] \geq 0$ for an arbitrary set of distinct points $x_0, x_1, \ \ , x_k$ belonging to $\mathbb{R}^+$. Writing $K_n(t) = \frac{n^n}{\Gamma(n)} t^n e^{-nt}$, it is easily seen that

$$(M_{n,x}[f])[x_0, \ \ , x_k]$$

$$= \left(\int_0^\infty f(tx) K_n(t) dt \right) [x_0, . \ \ , x_k]$$

$$= \int_0^\infty \left(f(tx) [x_0, .. \ , x_k] \right) K_n(t) \, dt$$

$$= \int_0^\infty f \, [tx_0, \ tx_1, \ \ , tx_k] \ K_n(t) dt \, .$$

Now, since $K_n(t) \geq 0$, the k-convexity of f implies that $(M_{n,x}[f])[x_0, .. , x_k] \geq 0$. This completes the proof.

A real valued function f is said to be <u>starshaped</u> on an interval I, if $f(\alpha x) \leq \alpha f(x)$ for every $\alpha \in [0,1]$, $x \in I$. A study of such functions is made in Beckenbach [2], Bruckner and Ostrow [8] and Popoviciu [61] Lupaş [49] established that the Bernstein polynomials map a starshaped function into a starshaped function. The Post-Widder operators $M_{n,x}$ also possess this property

THEOREM 1.2.3 Let S denote the class of all starshaped functions on $(0,\infty)$. Then, if $f \in S$ and $M_{n,x}[f]$ exists for all $x \in (0,\infty)$, also $M_{n,x}[f] \in S$.

PROOF . The proof of the starshape preserving property for the operators $M_{n,x}$ turns out to be rather simple. For, let $f \in S$ and $\alpha \in [0,1]$. Then,

$$M_{n,\alpha x}[f(t)] = \frac{n^n}{\Gamma(n)} \int_0^\infty f(t\alpha x)\, t^n e^{-nt}\, dt$$

$$\leq \frac{n^n}{\Gamma(n)} \int_0^\infty \alpha f(tx)\, t^n e^{-nt}\, dt$$

$$= \alpha\, M_{n,x}[f(t)],$$

and hence $M_{n,x}[f] \in S$.

We close this section with the following remark A similar study, as done in the thesis, can also be carried out for the more general operators $M_{n,p,x}$, defined by

$$M_{n,p,x}[f(t)] = \frac{1}{\Gamma(n+p+1)} \left(\frac{n}{x}\right)^{n+p+1} \int_0^\infty f(t) t^{n+p} e^{-nt/x}\, dt,$$

where p is a real number Indeed, when $p = 0$, the operator $M_{n,p,x}$ reduces to the Post-Widder operator Also, it is interesting to note that amongst all the operators $M_{n,p,x}$, the Post-Widder operators are the only ones that possess the <u>area preserving property</u>. For, it is easily seen that

$$\int_0^\infty M_{n,p,x}[f]\, dx$$

$$= \frac{n}{n+p} \int_0^\infty f(t)\, dt$$

$$= \int_0^\infty f(x)\, dx,$$

if, and only if $p = 0$.

1.3 SOME EVALUATIONS AND A RECURRENCE FORMULA FOR MOMENTS

Using the definition of the operator $M_{n,x}$, one can easily obtain the following evaluations .

(1.6) $$M_{n,x}[t^k] = \frac{\Gamma(n+k+1)}{\Gamma(n+1)n^k}\, x^k,$$

for every real number k satisfying $n > -(k+1)$ From (1 6) it is evident that $M_{n,x}[t^k] \to x^k$ as $n \to \infty$.

The k-th moment $\mu_{n,k}(x)$ $(k = 0, 1, 2, \ldots)$ of the operator $M_{n,x}$ is defined by

$$\mu_{n,k}(x) = M_{n,x}[(t-x)^k].$$

From (1.6) it follows that

$$(1.7) \qquad \mu_{n,k}(x) = x^k \sum_{p=0}^{k} \binom{k}{p} (-1)^{k-p} \frac{\Gamma(n+p+1)}{\Gamma(n+1)n^p}, \quad (n > -(k+1)),$$

$$(1.8) \qquad \begin{cases} M_{n,x}[1] = 1, \\ M_{n,x}[t] = x + x/n, \\ M_{n,x}[t^2] = x^2\left(1 + \frac{3}{n} + \frac{2}{n^2}\right) \text{ and} \\ M_{n,x}[t^{-1}] = x^{-1}, \end{cases}$$

and

$$(1.9) \qquad \begin{cases} \mu_{n,0}(x) = 1, \\ \mu_{n,1}(x) = \frac{x}{n} \text{ and} \\ \mu_{n,2}(x) = x^2\left(\frac{1}{n} + \frac{2}{n^2}\right). \end{cases}$$

Also, for $n > \alpha x$, α a real number, we have

$$(1.10) \qquad M_{n,x}[e^{\alpha t}] = \left(1 - \frac{\alpha x}{n}\right)^{-(n+1)} \quad (\to e^{\alpha x},\ n \to \infty).$$

We now obtain a recurrence formula for the moments $\mu_{n,k}(x)$ in

<u>LEMMA 1.3.1</u> For $k = 1, 2, \ldots,$ we have

(1.11) $\mu_{n,k+1}(x) = \frac{x}{n}\{(k+1)\,\mu_{n,k}(x) + kx\,\mu_{n,k-1}(x)\}$.

<u>PROOF</u> From (1.7), for $k = 1, 2, \ldots,$ we have

$$x \frac{\partial}{\partial x} \mu_{n,k+1}(x) = (k+1)\,\mu_{n,k+1}(x).$$

By definition the left hand side is

$$= x \frac{\partial}{\partial x} \left\{ \frac{n^{n+1}}{\Gamma(n+1) x^{n+1}} \int_0^\infty (t-x)^{k+1}\, t^n\, e^{-nt/x}\, dt \right\}$$

$$= -(n+1)\mu_{n,k+1}(x) - (k+1)x\mu_{n,k}(x)$$

$$+ \frac{n}{x} \frac{n^{n+1}}{\Gamma(n+1)x^{n+1}} \int_0^\infty (t-x)^{k+1}\, t\, t^n\, e^{-nt/x}\, dt$$

$$= -(n+1)\mu_{n,k+1}(x) - (k+1)x\,\mu_{n,k}(x) + \frac{n}{x}\mu_{n,k+2}(x) + n\,\mu_{n,k+1}(x)$$

Hence,

$$(k+1)\,\mu_{n,k+1}(x) + (n+1)\,\mu_{n,k+1}(x) + (k+1)x\,\mu_{n,k}(x) - \frac{n}{x}\mu_{n,k+2}(x) - n\,\mu_{n,k+1}(x) = 0,$$

i.e.,

$$(k+2)\,\mu_{n,k+1}(x) + (k+1)x\,\mu_{n,k}(x) - \frac{n}{x}\mu_{n,k+2}(x) = 0.$$

Replacing k by k-1 we have the recurrence formula (1.11).

1.4 <u>A FEW BASIC LEMMAS</u>

<u>LEMMA 1.4.1</u> : For $k = 0, 1, 2, \ldots,$ $x^{-k}\mu_{n,k}(x)$ does not depend on x and

(1.12) $\sigma_{n,k} = O(n^{-[\frac{k+1}{2}]}),$

where $\sigma_{n,k} = x^{-k}\,\mu_{n,k}(x)$ and $[\frac{k+1}{2}]$ denotes the integral part of $\frac{k+1}{2}$.

PROOF That $x^{-k}\,\mu_{n,k}(x)$ is independent of x is evident from (1.7) The formula (1.11) may be re-written as

$$\sigma_{n,k+1} = \frac{k}{n}\{(\frac{k+1}{k})\,\sigma_{n,k} + \sigma_{n,k-1}\}$$

The lemma is obviously true for $k = o,1,2$. Now, assuming it to be true for $k \leq m$, we have

$$\begin{aligned}\sigma_{n,m+1} &= \frac{m}{n}\{(\frac{m+1}{m})\,O(n^{-[\frac{m+1}{2}]}) + O(n^{-[\frac{m}{2}]})\}\\ &= \frac{m}{n}\{O(n^{-[\frac{m}{2}]})\}\\ &= O(n^{-[\frac{m+2}{2}]}), \quad n \to \infty,\end{aligned}$$

showing that the result is true for $m+1$ Hence the lemma follows by induction.

LEMMA 1.4.2 For $k = o,1,2,\ldots$, there exist constants $C_{k,o}, C_{k,1}, \ldots, C_{k,[k/2]}$ such that

$$\sigma_{n,k} = n^{-[\frac{k+1}{2}]}\sum_{p=o}^{[\frac{k}{2}]}\frac{C_{k,p}}{n^p}. \tag{1.13}$$

PROOF From (1.6), we have

$$\begin{aligned}x^{-p}\,M_{n,x}[t^p] &= \frac{\Gamma(n+p+1)}{\Gamma(n+1)n^p}\\ &= (1+\frac{p}{n})\,(1+\frac{p-1}{n}) \qquad (1+\frac{1}{n})\\ &= \sum_{i=o}^{p}\frac{d_{p,i}}{n^i},\end{aligned}$$

for suitable $d_{p,i}$'s

Hence,

$$\sigma_{n,k} = \sum_{p=o}^{k} (-1)^{k-p} \binom{k}{p} \sum_{i=o}^{p} \frac{d_{p,i}}{n^i}$$

$$= \sum_{j=o}^{k} \frac{a_{k,j}}{n^j} \text{, for suitable } a_{k,j}\text{'s}$$

But, by Lemma 1.4.1,

$$\sigma_{n,k} = O\,(n^{-[\frac{k+1}{2}]})$$

Therefore $a_{k,j} = o$ for $j = o, 1, \ldots, [k/2]$.

Hence,

$$\sigma_{n,k} = \sum_{j=[\frac{k+1}{2}]}^{k} \frac{a_{k,j}}{n^j}$$

$$= n^{-[\frac{k+1}{2}]} \sum_{p=o}^{[\frac{k}{2}]} \frac{C_{k,p}}{n^p},$$

where $C_{k,p} = a_{k,p+[\frac{k+1}{2}]}$.

This completes the proof of the lemma.

Let $x > \delta > o$ and $I_\delta(x) = (x-\delta, x+\delta)$ We denote the set $(o,\infty)\setminus I_\delta(x)$ by $J_\delta(x)$.

<u>LEMMA 1.4.3</u> . If $f \in B^*$ then for an arbitrary $k > o$,

$$(1.14) \qquad \frac{n^{n+1}}{\Gamma(n+1)x^{n+1}} \int_{J_\delta(x)} f(t)\, t^n e^{-nt/x}\, dt = o(n^{-k}),\ n \to \infty$$

Also, if $o < \delta < a < b < \infty$, then (1.14) holds uniformly in $x \in [a,b]$.

<u>PROOF</u> . For $t \in J_\delta(x)$, it is easy to see that

$$\sup_{t\in J_\delta(x)} \left(\frac{t}{x} e^{-t/x}\right)^n = \left(\frac{x+\delta}{x} e^{-\frac{x+\delta}{x}}\right)^n.$$

Hence, for two arbitrary numbers $\delta_o, A_o > o$, $o < \delta_o < x-\delta < A_o < \infty$, we have

$$\left| \frac{n^{n+1}}{\Gamma(n+1)x^{n+1}} \int_{[\delta_o, A_o]\setminus I_\delta(x)} f(t)\, t^n e^{-nt/x}\, dt \right|$$

$$\leq \frac{n^n}{\Gamma(n)} \int_{[\delta_o, A_o]\setminus I_\delta(x)} |f(t)x^{-1}| \left(\frac{x+\delta}{x} e^{-\frac{x+\delta}{x}}\right)^n dt$$

$$\leq \frac{n^n}{\Gamma(n)} \left(\frac{x+\delta}{x} e^{-\frac{x+\delta}{x}}\right)^n B,$$

where $B = \int_{[\delta_o, A_o]\setminus I_\delta(x)} |f(t)x^{-1}|\,dt$.

By Stirling's formula

$$\frac{n^n}{\Gamma(n)} \left(\frac{x+\delta}{x} e^{-\frac{x+\delta}{x}}\right)^n \cong \sqrt{n/2\pi} \left(\frac{x+\delta}{x} e^{-\delta/x}\right)^n,$$

and it is easy to see that for some $\epsilon > o$, $\frac{x+\delta}{x} e^{-\delta/x} < 1-\epsilon$ (also uniformly in $x \in [a,b]$). It follows that

$$\frac{n^n}{\Gamma(n+1)x^{n+1}} \int_{[\delta_o, A_o]\setminus I_\delta(x)} f(t) t^n e^{-\frac{nt}{x}}\, dt = o(n^{-k}), n\to\infty, \tag{1.15}$$

for an arbitrary $k > o$ Further, (1.15) holds uniformly in $x \in [a,b]$ provided δ_o, A_o are fixed and $o < \delta_o < a < b < A_o < \infty$. Now, since $f \in \bar{B}^*$ there exist m, $B_m, b_m > o$ such that

$$|f(t)| < B_m t^{-m}, \quad o < t < b_m .$$

Let us choose $\delta_o > o$ such that for some $1 > \epsilon > o$, $\delta_o < \min \{b_m, \frac{x(1-\epsilon}{e}$ (for $x \in [a,b]$ we choose $\delta_o < \min \{ b_m, \frac{a(1-\epsilon)}{e} \}$).

Then,

$$\left| \frac{n^{n+1}}{\Gamma(n+1)x^{n+1}} \int_0^{\delta_0} f(t)t^n e^{-nt/x} dt \right|$$

$$\leq B_m \frac{n^{n+1}}{\Gamma(n+1)x^{n+1}} \int_0^{\delta_0} t^{n-m} dt$$

$$= B_m \frac{n^{n+1}}{\Gamma(n+1)x^{n+1}} \frac{\delta_0^{n-m+1}}{(n-m+1)}$$

Since $\frac{n^{n+1}}{\Gamma(n+1)x} \left(\frac{\delta_0}{x}\right)^n \simeq \frac{1}{\sqrt{2\pi n}\ x} \left(\frac{e\delta_0}{x}\right)^n$

and by the assumption on δ_0, $\frac{e\delta_0}{x} < 1-\epsilon$,

we have

$$\frac{n^{n+1}}{\Gamma(n+1)x^{n+1}} \int_0^{\delta_0} f(t)t^n e^{-\frac{nt}{x}} dt = o(n^{-k}),\ n \to \infty.$$

(and uniformly for $x \in [a,b]$). It follows that

$$(1.16) \qquad \frac{n^{n+1}}{\Gamma(n+1)x^{n+1}} \int_0^{\delta_0} f(t)t^n e^{-nt/x} dt = o(n^{-k}),\ n \to \infty,$$

for each $k > 0$ (and uniformly for $x \in [a,b]$ in the uniformity case). Next, since $f \in B^*$ there exist $p, A_p, a_p > 0$ such that

$$|f(t)| < A_p e^{pt}, \quad t > a_p.$$

Now, it is clear that if we choose a number $q_0 > 0$ such that for some $\epsilon > 0$, $q_0 < \frac{1}{x} - \epsilon$ (in the uniformity case $q_0 < \frac{1}{b} - \epsilon$) then there exists a $t_0 > 0$ such that for all $t > t_0$, $t < e^{q_0 t}$.

If we choose $A_0 > \max\{a_p,\ t_0,\ \frac{1}{\epsilon}\log\left(\frac{e}{x}\right)\}$

(in the uniformity case $A_0 > \max\{a_p, t_0, \frac{1}{\epsilon}\log\left(\frac{e}{a}\right)\}$) then

$$\left| \frac{n^{n+1}}{\Gamma(n+1)x^{n+1}} \int_{A_o}^{\infty} f(t)t^n e^{-nt/x} dt \right|$$

$$\leq \frac{A_p n^{n+1}}{\Gamma(n+1)x^{n+1}} \int_{A_o}^{\infty} \exp\{ t p - \frac{nt}{x} + q_o nt \} dt$$

$$= \frac{A_p n^{n+1}}{\Gamma(n+1)x^{n+1}} \int_{A_o}^{\infty} \exp t \{p - n(\frac{1}{x} - q_o)\} dt$$

$$\leq \frac{A_p n^{n+1}}{\Gamma(n+1)x^{n+1}} \int_{A_o}^{\infty} \exp t \{p - n\epsilon\} dt$$

$$= \frac{A_p n^{n+1}}{\Gamma(n+1)x^{n+1}} \frac{\exp A_o \{ p - n\epsilon \}}{(n\epsilon - p)}$$

$$\cong A_p \frac{1}{\sqrt{2\pi n}\ x} (\frac{e}{x})^n \frac{\exp A_o \{p - n\epsilon\}}{\epsilon},$$

since $\frac{e^{1-\epsilon A_o}}{x} < \frac{e^{1-(\epsilon/\epsilon)\log(e/x)}}{x} = 1$ (and in the uniformity case $\frac{e^{1-\epsilon A_o}}{x} < \frac{a}{x} \leq 1$, $x \in [a,b]$).

It follows that

$$(1.17) \qquad \frac{n^{n+1}}{\Gamma(n+1)x^{n+1}} \int_{A_o}^{\infty} f(t)t^n e^{-nt/x} dt = o(n^{-k}),\ n \to \infty,$$

for each $k > o$, and that (1.17) holds uniformly in $x \in [a,b]$ in the uniformity case.

It follows from (1.15),(1.16) and (1.17) with δ_o, A_o as in (1.16), (1.17), that

$$\frac{n^{n+1}}{\Gamma(n+1)x^{n+1}} \int_{J_\delta(x)} f(t)t^n e^{-nt/x} dt = o(n^{-k}),\ n \to \infty,$$

for each $k > o$, and that this holds uniformly in $x \in [a,b]$ for each fixed interval $[a,b]$, with $o < \delta < a < b < \infty$. This completes the proof of the lemma.

1.5 BASIC CONVERGENCE, ERROR ESTIMATES AND ASYMPTOTIC FORMULAE

The following theorem, which is a simple consequence of Lemma 1.4.3, establishes the basic convergence of the operators $M_{n,x}$ for $f \in B^*$.

THEOREM 1.5.1. If $f \in B^*$ and is continuous at a point $x \in (o,\infty)$, then

$$(1.18) \qquad \lim_{n \to \infty} M_{n,x}[f(t)] = f(x).$$

Further, if $f(x)$ is continuous on $[a,b] \subset (o,\infty)$, then (1.18) holds uniformly for all $x \in [a,b]$.

PROOF Since f is continuous at x, for an arbitrary $\epsilon > o$ there exists a $\delta > o$ such that $|f(t)-f(x)| < \epsilon$ whenever $|t-x| < \delta$. Let $I_\delta(x)$ and $J_\delta(x)$ be as in Lemma 1.4.3. Then for all $t \in (o,\infty)$ we can write,

$$|f(t)-f(x)| < \epsilon + \{ |f(t)| + |f(x)| \} \chi_{J_\delta(x)}.$$

Further, if $f(x)$ is continuous in $[a,b]$ then the above 'δ' can be chosen independent of $x \in [a,b]$ and therefore the last inequality holds for each $x \in [a,b]$. Operating on this inequality by the operator $M_{n,x}$, we get

$$|M_{n,x}[f(t)] - f(x)| < \epsilon + M_{n,x}[\{ |f(t)| + |f(x)| \} \chi_{J_\delta(x)}],$$

which by Lemma 1.4.3 and due to the arbitrariness of ϵ yields (1.18). Hence the proof is complete.

THEOREM 1.5.2 If $f \in B^*$ and is continuous on $\mathbb{R}^+$, then for all $x \in \mathbb{R}^+$,

(1.19) $$\left| M_{n,x}[f(t)] - f(x) \right| \leq \{1 + x^2(1 + \frac{2}{n})\}\, \omega(f, n^{-1/2})$$

where $\omega(f,\delta)$ denotes the modulus of continuity of f.

Another estimate is given by

(1.20) $$\left| M_{n,x}[f(t)] - f(x) \right| \leq \{1 + x\sqrt{1+\frac{2}{n}}\}\, \omega(f, n^{-1/2}).$$

PROOF . We have

(1.21) $$|M_{n,x}[f(t)] - f(x)| \leq M_{n,x}[\,|f(t)-f(x)|\,]$$

But, for $x, t \in \mathbb{R}^+$, $|f(t)-f(x)| \leq \omega(f, |t-x|)$

(1.22) $$= \omega(f, \frac{|t-x|}{n^{-1/2}} n^{-1/2})$$

$$\leq \{1 + \frac{|t-x|}{n^{-1/2}}\}\, \omega(f, n^{-1/2}).$$

Therefore, since $M_{n,x}[1] = 1$,

$$|M_{n,x}[f(t)] - f(x)| \leq M_{n,x}[\{1 + \frac{|t-x|}{n^{-1/2}}\}\, \omega(f, n^{-1/2})]$$

$$\leq \{1 + M_{n,x}[\frac{|t-x|}{n^{-1/2}}]\}\, \omega(f, n^{-1/2})$$

$$\leq \{1 + \frac{\sqrt{M_{n,x}[(t-x)^2]}}{n^{-1/2}}\}\, \omega(f, n^{-1/2}),$$

(by Schwarz's inequality)

$$= \{1 + x\sqrt{1+2/n}\}\, \omega(f, n^{-1/2}).$$

This establishes the estimate (1.20)

In a similar manner, we also have

$$|M_{n,x}[f(t)] - f(x)| \le \{1+M_{n,x}[\frac{(t-x)^2}{n^{-1}}]\}\omega(f,n^{-1/2})$$

$$= \{1+x^2(1+2/n)\}\,\omega(f,n^{-1/2}),$$

which is the same as (1.19). This completes the proof of the theorem.

<u>THEOREM 1.5.3</u> . If $f \in B^*$ and f' exists and is continuous on $\mathbb{R}^+$, then for all $x \in \mathbb{R}^+$,

$$(1.23)\quad |M_{n,x}[f(t)] - f(x)| \le \frac{x|f'(x)|}{n} + \{\frac{x}{\sqrt{n}}\sqrt{1+2/n} + x^2(\frac{1}{\sqrt{n}} + \frac{1}{n^{3/2}})\}\omega(f',n^{-1/2}).$$

<u>PROOF</u> By the assumption on f', for all t , $x \in \mathbb{R}^+$, we have

$$f(t) = f(x)+(t-x)f'(x) + (t-x)\{f'(\eta) - f'(x)\},$$

where η lies between t and x. Therefore,

$$(1.24)\quad \begin{aligned}|f(t)-f(x)-(t-x)f'(x)| &\le |t-x|\,|f'(\eta) - f'(x)| \\ &\le |t-x|\,(1 + \frac{|t-x|}{n^{-1/2}})\,\omega(f',n^{-1/2}).\end{aligned}$$

Operating on this inequality by $M_{n,x}$, we get

$$\begin{aligned}|M_{n,x}[f(t)] - f(x)| &\le \frac{x|f'(x)|}{n} + \{M_{n,x}[|t-x|(1+\frac{|t-x|}{n^{-1/2}})]\}\,\omega(f',n^{-1/2}) \\ &= \frac{x|f'(x)|}{n} + \{M_{n,x}[|t-x| + \frac{(t-x)^2}{n^{-1/2}}]\}\,\omega(f',n^{-1/2}) \\ &= \frac{x|f'(x)|}{n} + \{\frac{x}{\sqrt{n}}\sqrt{1+2/n} + \frac{x^2}{\sqrt{n}}(1+\frac{2}{n})\}\,\omega(f',n^{-1/2}) \\ &= \le \frac{x|f'(x)|}{n} + \{\frac{x}{\sqrt{n}}\sqrt{1+2/n} + x^2(\frac{1}{\sqrt{n}} + \frac{2}{n^{3/2}})\}\,\omega(f',n^{-1/2})\end{aligned}$$

The following theorem determines an asymptotic formula of Voronovskaja type for the operators $M_{n,x}$. We denote by $< a,b >$ an open interval containing $[a,b]$.

<u>THEOREM 1.5.4</u>. If $f \in B^*$ and f'' exists at some $x \in \mathbb{R}^+$, then

$$(1.25) \qquad M_{n,x}[f(t)] - f(x) = \frac{1}{n}[2xf'(x)+f''(x)x] + o(n^{-1}), n \to \infty .$$

Further, if f'' exists in $< a,b > \subset \mathbb{R}^+$ and is continuous on $[a,b]$ then (1.25) holds uniformly in $x \in [a,b]$

<u>PROOF</u> Under the assumption on f, we have

$$(1.26) \qquad f(t) = \sum_{k=o}^{2} \frac{f^{(k)}(x)}{k!}(t-x)^k + h(t;x),$$

where $h(t,x) \in B^*$ and is such that given an arbitrary $\epsilon > o$ there exists a $\delta > o$ such that

$$(1.27) \qquad |h(t,x)| \leq \epsilon(t-x)^2, \quad |t-x| < \delta.$$

Operating the inequality (1.27) by $M_{n,x}$ and using Lemma 1.4.3, we get

$$|M_{n,x}[h(t,x)]| \leq \epsilon\, M_{n,x}[(t-x)^2] + M_{n,x}[|h(t,x)|\chi_{J_\delta}(x)(t)]$$

$$(1.28) \qquad = \epsilon\left(\frac{x^2}{n} + \frac{2x^2}{n^2}\right) + o(n^{-1}), \quad n \to \infty .$$

Further, if f'' exists in $< a,b >$ and is continuous on $[a,b]$, then δ in (1.27) could be chosen to be independent of x. Hence in this case, the o-term in (1.28) holds uniformly in $x \in [a,b]$. It follows, due to the arbitrariness of ϵ, that

$$(1.29) \qquad M_{n,x}[h(t,x)] = o(n^{-1}), \quad n \to \infty .$$

Further, in the above mentioned uniformity case (1.29) holds uniformly in $x \in [a,b]$.

Hence from (1.26),

$$M_{n,x}[f(t)] = f(x) + \frac{f'(x)x}{n} + \frac{f''(x)}{2}\left[\frac{x^2}{n} + \frac{2x^2}{n^2}\right] + o(n^{-1}), n \to \infty$$

$$= f(x) + \frac{1}{n}\left[2xf'(x) + x^2 f''(x)\right] + o(n^{-1}),\ n \to \infty,$$

and in the uniformity case, this relation holds uniformly in $x \in [a,b]$. This completes the proof

The following result generalises the above theorem and will be used in the theory of linear combinations of the operators $M_{n,x}$ in section 1 6.

THEOREM 1.5.5. If $f \in B^*$ and $f^{(k)}$ exists $(k = 1,2,.\ )$ at a point $x \in \mathbb{R}^+$, then

$$(1.30) \qquad M_{n,x}[f(t)] = \sum_{p=o}^{k} \frac{f^{(p)}(x)}{p!} \mu_{n,p}(x) + o(n^{-k/2}),\ n \to \infty .$$

Further, if $f^{(k)}$ exists in $\langle a,b \rangle \subset \mathbb{R}^+$ and is continuous at each $x \in [a,b]$ then (1.30) holds uniformly in $x \in [a,b]$.

PROOF : The theorem can be easily proved along the lines of the proof of Theorem 1.5.4.

1.6 LINEAR COMBINATIONS

The m-th linear combination of the operators $M_{n,x}$, denoted by $M_{n,x}^{[m]}$ is defined by

$$(1.31) \quad M_{n,x}^{[m]}[f(t)] = \frac{1}{\Delta_m}\begin{vmatrix} M_{\alpha_0 n,x}[f(t)] & \frac{1}{\alpha_0} & \frac{1}{\alpha_0^2} & \cdots & \frac{1}{\alpha_0^{m-1}} \\ M_{\alpha_1 n,x}[f(t)] & \frac{1}{\alpha_1} & \frac{1}{\alpha_1^2} & \cdot & \frac{1}{\alpha_1^{m-1}} \\ & & \cdot & & \\ \cdot & & \cdots & \cdots & \\ M_{\alpha_{m-1} n,x}[f(t)] & \frac{1}{\alpha_{m-1}} & \frac{1}{\alpha_{m-1}^2} & & \frac{1}{\alpha_{m-1}^{m-1}} \end{vmatrix},$$

where α_i's are certain fixed positive real numbers and Δ_m is the determinant obtained by replacing the elements in the first column of the above determinant by the entires 1 Linear combinations of this type were introduced by Rathore [67]

<u>THEOREM 1.6.1</u> . If $f \in B^*$ and $f^{(2k)}$ $(k = 1,2,\ldots,)$ exists at a point $x \in \mathbb{R}^+$, then

$$(1.32) \quad M_{n,x}^{[k]}[f(t)] - f(x) = O(n^{-k}), \; n \to \infty,$$

and

$$(1.33) \quad M_{n,x}^{[k+1]}[f(t)] - f(x) = o(n^{-k}), \; n \to \infty .$$

Further, if $f^{(2k)}$ exists in $\langle a,b \rangle \subset \mathbb{R}^+$ and is continuous at each $x \in [a,b]$ then (1.32) - (1.33) hold uniformly for each $x \in [a,b]$.

<u>PROOF</u> If $f^{(2k)}$ (x) exists, then by Theorem 1.5.5,

$$M_{n,x}[f(t)] - f(x) = \sum_{r=1}^{2k} \frac{f^{(r)}(x)}{r!} \mu_{n,r}(x) + o(n^{-k}), \; n \to \infty,$$

and this holds uniformly in the case when $f^{(2k)}(x)$ exists in $\langle a,b \rangle$ and is continuous on $[a,b]$. Therefore,

$$(1.34)\quad M_{n,x}^{[k+1]}[f(t)]-f(x)$$

$$= \frac{1}{\Delta_{k+1}} \begin{vmatrix} \sum_{r=1}^{2k} \frac{f^{(r)}(x)}{r!}\mu_{\alpha_0 n,r}(x) & \frac{1}{\alpha_0} & \frac{1}{\alpha_0^2} & \cdots & \frac{1}{\alpha_0^k} \\ \sum_{r=1}^{2k} \frac{f^{(r)}(x)}{r!}\mu_{\alpha_1 n,r}(x) & \frac{1}{\alpha_1} & \frac{1}{\alpha_1^2} & \cdots & \frac{1}{\alpha_1^k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sum_{r=1}^{2k} \frac{f^{(r)}(x)}{r!}\mu_{\alpha_k n,r}(x) & \frac{1}{\alpha_k} & \frac{1}{\alpha_k^2} & \cdots & \frac{1}{\alpha^k} \end{vmatrix}$$

$$+\ o(n^{-k}),\ n \to \infty,$$

and that (1.34) holds uniformly in the above mentioned uniformity case. Also, from Lemma 1.4.2,

$$\mu_{n,r} = x^r n^{-[\frac{r+1}{2}]} \sum_{p=o}^{[\frac{r}{2}]} \frac{C_{r,p}}{n^p}.$$ Putting the values of the moments $\mu_{\alpha_0 n,r}$, $\mu_{\alpha_1 n,r}$, ..., $\mu_{\alpha_k n,r}$ in (1.34), we get (1.33). From the above proof it can be seen that (1.32) also holds. Further, (1.32)-(1.33) hold uniformly in the uniformity case. This completes the proof of the theorem.

<u>THEOREM 1.6.2</u> : If $f \in B^*$ and $f^{(2k)}$ exists and is continuous on $\langle a,b\rangle \subset \mathbb{R}^+$, then for all $x \in [a,b]$,

$$(1.35)\quad |M_{n,x}^{[k+1]}[f(t)]-f(x)| \le \max\{\frac{C}{n^k}\,\omega(f^{(2k)};n^{-1/2}),\ \frac{C'}{n^{k+1}}\},$$

where C is a constant depending on k, C' is a constant depending on k and f both and $\omega(f^{(2k)};\delta)$ denotes the local modulus of continuity of $f^{(2k)}$ on $\langle a,b\rangle$.

PROOF   With the hypothesis on f, for all $x \in [a,b]$, we can write

$$(1.36)\quad f(t) = \sum_{p=0}^{2k} \frac{f^{(p)}(x)}{p!}(t-x)^p + \frac{(t-x)^{2k}}{(2k)!}\,[f^{(2k)}(\eta)-f^{(2k)}(x)]\,\chi_{\langle a,b\rangle}(t)$$
$$+\ h(t,x),\quad t>0,$$

where $\chi_{\langle a,b\rangle}(t)$ denotes the characteristic function of $\langle a,b\rangle$, η lies between t and x and h(t,x) is a certain function belonging to B^* such that it vanishes for $t \in \langle a,b\rangle$. Using the definition of $M_{n,x}^{[k+1]}[f(t)]$ and (1.36), for some constants B_j, $j = 0,1,\ldots,k$ we have

$$M_{n,x}^{[k+1]}[f(t)] - f(x) = \sum_{j=0}^{k} \{ B_j\,[M_{\alpha_j n,x}[f(t)] - f(x)]\},$$

$$(\text{since } B_0 + \ \ , B_k = 1)$$

$$= \sum_{j=0}^{k} \{ B_j (M_{\alpha_j n,x}[\sum_{p=0}^{2k} \frac{f^{(p)}(x)}{p!}(t-x)^p + \frac{(t-x)^{2k}}{(2k)!}(f^{(2k)}(\eta)-f^{(2k)}(x))$$
$$\chi_{\langle a,b\rangle}(t) + h(t,x)] - f(x))\}.$$

Thus, defining

$$T_{x,\alpha_j n}(t) = \left(\frac{\alpha_j n}{x}\right)^{\alpha_j n+1} \frac{1}{\Gamma(\alpha_j n+1)}\; t^{\alpha_j n}\, e^{-\alpha_j n t/x},$$

we have

$$M_{n,x}^{[k+1]}[f(t)] - f(x) = \{ M_{n,x}^{[k+1]}[\sum_{p=0}^{2k} \frac{f^{(p)}(x)}{p!}(t-x)^p] - f(x)\}$$

$$+ \sum_{j=0}^{k} B_j \int_0^\infty \frac{(t-x)^{2k}}{(2k)!}(f^{(2k)}(\eta)-f^{(2k)}(x))\,T_{x,\alpha_j n}(t)\ \chi_{\langle a,b\rangle}(t)\,dt$$

$$(1.37)\qquad + \sum_{j=0}^{k} B_j \int_0^\infty h(t,x)\ T_{x,\alpha_j n}(t)\,dt$$

$$= \Sigma + \Sigma + \Sigma\text{, say}$$

In view of Theorem 1.6.1 (we use 1.33 with k replaced by k+1), it is clear that

(1.38) $\quad |\Sigma_1| \leq C_1 n^{-k-1}$,

where C_1 depends on the maximum modulii of various derivatives $f^{(p)}(x)$ on $[a,b]$ and is independent of n. To evaluate Σ_2 we proceed as follows :

$$\int_0^\infty \frac{(t-x)^{2k}}{(2k)!} |f^{(2k)}(n) - f^{(2k)}(x)| \, \chi_{\langle a,b\rangle}(t)\, T_{x,\alpha_j n}(t)\, dt$$

$$\leq \frac{\omega(f^{(2k)};\delta)}{(2k)!} \left\{ \int_0^\infty (t-x)^{2k}\, T_{x,\alpha_j n}(t)dt + \frac{1}{\delta} \int_0^\infty |t-x|^{2k+1}\, T_{x,\alpha_j n}(t)dt \right\}.$$

The last expression does not exceed

$$\frac{\omega(f^{(2k)};\delta)}{(2k)!} \left\{ \frac{A_k}{(\alpha_j n)^k} + \frac{A_k'}{\delta(\alpha_j n)^{k+1/2}} \right\},$$

where A_k and A_k' are constants independent of n.

Hence,

$$|\Sigma_2| \leq \frac{\omega(f^{(2k)};\delta)}{(2k)!} \sum_{j=o}^{k} |B_j| \left(\frac{A_k}{(\alpha_j n)^k} + \frac{A_k'}{(\alpha_j n)^{k+1/2}} \right),$$

and with $\delta = n^{-1/2}$, we have

(1.39) $\quad |\Sigma_2| \leq \dfrac{C_2}{n^k}\, \omega(f^{(2k)}; n^{-1/2}).$

Finally, since $h(t;x)$ vanishes for $t \in \langle a,b\rangle$, by lemma 1.4.3 we have

$$|\Sigma_3| \leq \sum_{j=o}^{k} |B_j| \int_0^\infty T_{x,\alpha_j n}(t)|\, h(t;x)|\, dt$$

$$= O(n^{-(k+1)}),$$

uniformly in $x \in [a,b]$.

It follows that

$$(1.40) \quad |\Sigma_3| \leq \frac{C_3}{n^{k+1}},$$

for some constant C_3 independent of x and n. The estimates (1.38) - (1.40) prove the theorem.

COROLLARY 1.6.1 If $f \in B^*$ and $f^{(2k)}$ exists and belongs to Lip α, $0 < \alpha \leq 1$ on the interval $\langle a, b \rangle$, then

$$(1.41) \quad |M_{n,x}^{[k+1]}[f(t)] - f(x)| \leq M\, n^{-k-\alpha/2}, \; x \in [a,b]$$

where M is a constant independent of x and n.

Similarly, we can prove the following result for odd derivatives of f.

THEOREM 1.6.3 : If $f \in B^*$ and if $f^{(2k+1)}$ exists and is continuous on $\langle a, b \rangle$, then for all $x \in [a,b]$,

$$(1.42) \quad M_{n,x}^{[k+1]}[f(t)] - f(x)| \leq \{ \frac{C}{n^{k+1/2}} \omega(f^{(2k+1)}, n^{-1/2}), \frac{C'}{n^{k+1}} \},$$

where C is a constant depending on k, C' is a constant depending on k and f and $\omega(f^{(2k+1)}, \delta)$ denotes the local modulus of continuity of $f^{(2k+1)}$ on $\langle a, b \rangle$.

COROLLARY 1.6.2 If $f \in B^*$ and $f^{(2k+1)}$ exists and belongs to Lip α, $0 < \alpha \leq 1$ on the interval $\langle a, b \rangle$, then

$$(1.43) \quad |M_{n,x}^{[k+1]}[f(t)] - f(x)| \leq M\, n^{-k-\frac{1}{2}(1+\alpha)}, x \in [a,b]$$

where M is a constant independent of x and n.

CHAPTER II

BEST ASYMPTOTIC CONSTANTS IN ERROR ESTIMATES

2.1 INTRODUCTION

Let $\{L_n\}$, $n = 1,2,\ldots$ be a sequence of operators defined on a domain D of functions. Let D' be any subclass of D. If there exists a numerical sequence $\theta_n \to 0$ as $n \to \infty$ such that

$$(2.1) \qquad \lim_{n \to \infty} \theta_n^{-1} \sup_{f \in D'} |L_n(f,x)-f(x)| = C_A(x),$$

where $C_A(x)$ is a positive number, then $C_A(x)$ is called the Nikolskiǐ constant corresponding to the order θ_n of approximation of the class D' by the operators L_n. The Nikolskiǐ constant corresponding to D', a Lipschitz class, is termed a Lipschitz-Nikolskiǐ constant.

Let $\omega_f(\delta)$, as usual, denote the modulus of continuity of f. If

$$(2.2) \qquad \lim_{n \to \infty} \sup_{f} \frac{|L_n(f,x) - f(x)|}{\omega_f(\psi_n)} = C(x) > 0,$$

where $\psi_n \to 0$ as $n \to \infty$, then $C(x)$ is called the best asymptotic constant with modulus of continuity $\omega_f(\psi_n)$, for the operators L_n.

In this chapter, we obtain the Lipschitz-Nikolskiǐ constants and the best asymptotic constants with modulus of continuity, for the Post-Widder operators $M_{n,x}$. We also determine the best asymptotic constants in the case

when the argument 'ψ_n' in the modulus of continuity of the function f depends on x. It is interesting to note that with a proper choice of the argument '$\psi_n(x)$' the best constant turns out to be the same as the one obtained by Esseen [18] in the case of Bernstein Polynomials.

2.2 AN ESTIMATION

In order to obtain Lipschitz-Nikolskii constants for the operators $M_{n,x}$, we require an asymptotic evaluation of $M_{n,x}[|t-x|^{\alpha}]$, $\alpha > 0$. For this we proceed as follows. By the second equivalent form of the operators $M_{n,x}$, we have

$$M_{n,x}[|t-x|^{\alpha}] = \frac{n^n}{\Gamma(n)} \int_0^{\infty} |xt-x|^{\alpha} \; t^n \; e^{-nt} \, dt$$

$$= \frac{n^n}{\Gamma(n)} \; x^{\alpha} \int_0^{\infty} |t-1|^{\alpha} \; t^n \; e^{-nt} \, dt$$

$$= \frac{n^n}{\Gamma(n)} \; x^{\alpha} \; I, \qquad \text{say.}$$

For the values of $t \in \mathbb{R}^+$ satisfying $|t-1| < n^{-\gamma}$, where $1/3 < \gamma < 1/2$, we have $t = 1 + \theta n^{-\gamma}$, where θ lies between -1 and 1. We define

$$I_1 = \int_{1-n^{-\gamma}}^{1+n^{-\gamma}} |\theta|^{\alpha} \; n^{-\gamma\alpha} (1 + \theta n^{-\gamma})^n \; e^{-n(1+\theta n^{-\gamma})} \, dt,$$

where θ is related to t by $t = 1 + \theta n^{-\gamma}$ and put $I_2 = I - I_1$.

Thus

$$M_{n,x}[|t-x|^{\alpha}] = \frac{x^{\alpha} n^n}{\Gamma(n)} \{I_1 + I_2\}.$$

Now,

$$(1+\theta n^{-\gamma})^n = \exp n \left\{ \theta n^{-\gamma} - \frac{\theta^2 n^{-2\gamma}}{2} + \frac{\theta^3 n^{-3\gamma}}{3} - \qquad \right\}$$

$$= \exp\left\{ \theta n^{1-\gamma} - \frac{\theta^2}{2} n^{1-2\gamma} + o(1) \right\}$$

(uniformly in $\theta \in [-1,1]$). Therefore,

$$I_1 = \int_{-1}^{1} |\theta|^{\alpha} n^{-\gamma\alpha} e^{-n+o(1)} \exp\left\{ -\frac{\theta^2}{2} n^{1-2\gamma} \right\} n^{-\gamma} \, d\theta$$

$$= 2n^{-\gamma(\alpha+1)} e^{-n+o(1)} \int_0^1 |\theta|^{\alpha} \exp\left\{ -\frac{\theta^2}{2} n^{1-2\gamma} \right\} d\theta$$

Now, putting $\theta^2 n^{1-2\gamma} = 2t$, $\theta = n^{\gamma-1/2}\sqrt{2t}$, $d\theta = n^{\gamma-1/2} \frac{1}{\sqrt{2}} \frac{dt}{\sqrt{t}}$, we have

$$I_1 = \sqrt{2}\, n^{-\gamma(\alpha+1)} e^{-n+o(1)} 2^{\alpha/2} \int_0^{\frac{1}{2}n^{1-2\gamma}} n^{(1+\alpha)(\gamma-\frac{1}{2})} t^{\frac{\alpha-1}{2}} e^{-t} dt$$

$$= 2^{\alpha/2}\sqrt{2}\, n^{-(\frac{1+\alpha}{2})} e^{-n+o(1)} \int_0^{\frac{1}{2}n^{1-2\gamma}} t^{(\frac{\alpha+1}{2}-1)} e^{-t} dt$$

$$\cong 2^{\alpha/2}\sqrt{2}\, n^{-(\frac{1+\alpha}{2})} e^{-n} \Gamma\left(\frac{1+\alpha}{2}\right).$$

Thus

$$\frac{x^{\alpha} n^n}{\Gamma(n)} I_1 \cong \frac{2^{\alpha/2}\sqrt{2}\, x^{\alpha} n^{-(\frac{1+\alpha}{2})} n^n e^{-n} \Gamma(\frac{1+\alpha}{2})}{\sqrt{2\pi}\, n^{n-1/2} e^{-n}}$$

$$= \frac{\Gamma(\frac{1+\alpha}{2})}{\sqrt{\pi}} \left(\frac{2x^2}{n}\right)^{\alpha/2}.$$

Further, we have

$$I_2 = \frac{n^n x^\alpha}{\Gamma(n)} \int_{|t-1| \geq n^{-\gamma}} |t-1|^\alpha t^n e^{-nt} dt$$

$$\leq \frac{n^n x^\alpha}{\Gamma(n)} \int_0^\infty |t-1|^{2s} t^n e^{-nt} n^{\gamma(2s-\alpha)} dt,$$

(where s is any integer $> \alpha/2$)

$$= x^\alpha n^{\gamma(2s-\alpha)} M_{n,1} [(t-1)^{2s}]$$

$$= x^\alpha n^{\gamma(2s-\alpha)} O\left(\frac{1}{n^s}\right) \text{ (from evaluations of Chapter I)}$$

$$= o(n^{-\alpha/2}) \quad \left(\text{because } \gamma(2s-\alpha) - s < \tfrac{1}{2}(2s-\alpha)-s = -\alpha/2\right).$$

Hence

$$(2.3) \qquad M_{n,x}[|t-x|^\alpha] \cong \frac{\Gamma(\frac{1+\alpha}{2})}{\sqrt{\pi}} \left(\frac{2x^2}{n}\right)^{\alpha/2}$$

2.3 THE LIPSCHITZ-NIKOLSKIǏ CONSTANTS

In this section we use the asymptotic evaluation (2.3) to obtain Lipschitz-Nikolskiǐ constants for $M_{n,x}$.

THEOREM 2.3.1 If $E_n(\alpha, x) = \sup_f \{|M_{n,x}[f(t)] - f(x)|\}$, $(x > 0,$ where the supremum is taken over all functions of the class $\mathrm{Lip}_1 \alpha\ (0 < \alpha \leq 1)$, then

$$(2.4) \qquad \lim_{n \to \infty} n^{\alpha/2} E_n(\alpha, x) = \frac{\Gamma(\frac{1+\alpha}{2})}{\sqrt{\pi}} (2x^2)^{\alpha/2}.$$

PROOF If $f \in Lip_1 \alpha$, then $|f(t)-f(x)| \leq |t-x|^{\alpha}$, and also the function $g(t) \equiv |t-x|^{\alpha} \in Lip_1 \alpha$. Hence, $E_n(\alpha, x) = M_{n,x}[\,|t-x|^{\alpha}\,]$, using the linearity and positivity of $M_{n,x}$ and the fact that $M_{n,x}[1] = 1$. Therefore, by (2.3)

$$\lim_{n \to \infty} n^{\alpha/2} E_n(\alpha, x) = \lim_{n \to \infty} n^{\alpha/2} \frac{\Gamma(\frac{1+\alpha}{2})}{\sqrt{\pi}} \left(\frac{2x^2}{n}\right)^{\alpha/2}$$

$$= \frac{\Gamma(\frac{1+\alpha}{2})}{\sqrt{\pi}} (2x^2)^{\alpha/2}.$$

This completes the proof of the theorem.

THEOREM 2.3.2. If $E_n(\alpha, p, x) = \sup_f \{|\, M_{n,x}[f(t) - \sum_{k=0}^{p} \frac{(t-x)^k}{k!} f^{(k)}(x)]\,|\}$ $(x > 0,\ p = 1, 2, \ldots)$ where the supremum is taken over all functions with $f^{(p)} \in Lip_1 \alpha$ $(0 < \alpha \leq 1)$, then

$$\frac{2^{(p+3\alpha-2)/2}\,\Gamma(\frac{p+\alpha+1}{2})\, x^{p+\alpha}}{(1+\alpha)(2+\alpha)\ldots(p+\alpha)\sqrt{\pi}} \leq \underline{\lim}_{n \to \infty} n^{(p+\alpha)/2} E_n(\alpha, p, x)$$

(2.5)

$$\leq \overline{\lim}_{n \to \infty} n^{(p+\alpha)/2} E_n(\alpha, p, x) \leq \frac{2^{(p+\alpha)/2}\,\Gamma((p+\alpha+1)/2)\, x^{p+\alpha}}{(1+\alpha)(2+\alpha)\ldots(p+\alpha)\sqrt{\pi}}.$$

In order to prove the theorem we require the ollowing lemma from [67] .

LEMMA . The function $f(t) \equiv |t-x|^{\alpha}\ \mathrm{sgn}\,(t-x) \in Lip_{2^{1-\alpha}} \alpha$, where x is a fixed real number and $0 < \alpha \leq 1$.

PROOF OF THEOREM 2.3.2 . To prove the second inequality in (2.5), if $f^{(p)} \in Lip_1 \alpha$, we have

$$\left| f(t) - \sum_{k=o}^{p} \frac{(t-x)^k}{k!} f^{(k)}(x) \right| = \left| \int_x^t \quad \int_x^t [f^{(p)}(t)-f^{(p)}(x)] \quad dt \quad dt \right|$$

p- times

$$\leq \left| \int_x^t \cdot \int_x^t |f^{(p)}(t) - f^{(p)}(x)| \, dt \quad .dt \right|$$

$$\leq \left| \int_x^t \ldots \int_x^t | t-x |^{\alpha} dt \, . \quad dt \right|$$

$$= \frac{|t-x|^{\alpha+p}}{(1+\alpha)(2+\alpha). \quad (p+\alpha)},$$

where $\int_x^t$ for $x > t$ is interpreted as $-\int_t^x$. Therefore,

$$\overline{\lim_{n\to\infty}} \; n^{(\frac{p+\alpha}{2})} E_n(\alpha, p, x) \leq \overline{\lim_{n\to\infty}} \; \frac{n^{(\frac{p+\alpha}{2})} M_{n,x} [|t-x|^{\alpha+p}]}{(1+\alpha)(2+\alpha). \;.(p+\alpha)}$$

$$= \frac{2^{(\frac{p+\alpha}{2})} \Gamma((p+\alpha+1)/2) x^{p+\alpha}}{(1+\alpha)(2+\alpha) \quad (p+\alpha)}.$$

The first inequality in (2.5) follows from a consideration of the function $f(t) \equiv |t-x|^{p+\alpha}/(1+\alpha)(2+\alpha) \quad (p+\alpha)2^{1-\alpha}$ For, by the above lemma it easily follows that $f^{(p)} \in Lip_1 \alpha$. This completes the proof of the theorem.

<u>COROLLARY 2.3.1</u> We have

$$(2.6) \qquad \lim_{n\to\infty} n^{(\frac{p+1}{2})} E_n(1,p,x) = \frac{2^{(\frac{p+1}{2})} \Gamma(p/2+1) \; x^{p+1}}{(p+1)! \; \sqrt{\pi}}, \quad (x > 0).$$

<u>PROOF</u> . Putting $\alpha = 1$, the corollary follows from theorem 2.3.2.

2.4 AN IMPROVED ERROR ESTIMATE FOR CONTINUOUSLY DIFFERENTIABLE FUNCTIONS

In this section, we obtain an improvement over the error estimate of (1.23) for continuously differentiable functions. Let $f \in B^*$ and let f' exist and be continuous with the modulus of continuity $\omega_{f'}(\delta)$. Then, we have

$$|f(t)-f(x)-f'(x)(t-x)| = |\int_x^t (f'(t)-f'(x)dt|$$

$$\leq |\int_x^t |f'(t)-f'(x)|dt|$$

$$\leq |\int_x^t (1+\frac{|t-x|}{n^{-1/2}}) \omega_{f'}(n^{-1/2})dt|,$$

and therefore,

(2.7) $$|f(t)-f(x)-f'(x)(t-x)| \leq \{|t-x| + \frac{(t-x)^2}{2n^{-1/2}}\} \omega_{f'}(n^{-1/2}).$$

Operating on (2.7) by $M_{n,x}$ we get, by (2.3), the improved estimate

(2.8) $$|M_{n,x}[f(t)] - f(x)|$$

$$\leq \frac{x|f'(x)|}{n} + \{\frac{x}{\sqrt{n}} (\sqrt{2/\pi} + o(1)) + \frac{x^2}{2\sqrt{n}}(1 + \frac{2}{n})\} \omega_{f'}(n^{-1/2}).$$

In a similar manner the error estimate (1.20) can be improved to the following

(2.9) $$|M_{n,x} [f(t)] - f(x)| \leq \{1 + x(\sqrt{2/\pi} + o(1)\} \omega_f(n^{-1/2}),$$

where $f \in B^*$ is assumed to be continuous with the modulus of continuity $\omega_f(\delta)$.

Indeed in the next section, we shall further improve (2.9) by obtaining the best asymptotic constant with the modulus of continuity $\omega_f(\circ)$.

2.5 BEST ASYMPTOTIC CONSTANT WITH MODULUS OF CONTINUITY

The following theorem gives the best asymptotic constant with modulus of continuity, for the operators $M_{n,x}$, where, in addition to the dependence on n, the argument in the modulus of continuity also depends on x.

THEOREM 2.5.1 : Let ϕ be a positive function on (o,∞). If

$$(2.10) \qquad C_n(x) = \inf \{ C \quad |M_{n,x}[f(t)] - f(x)| \leq C\ \omega_f(\phi(x)n^{-1/2}),$$

$$\text{for all continuous } f \in B^* \},\ (x > 0),$$

then

$$(2.11) \qquad C_n(x) \cong C_\infty(x) = 2 \sum_{j=0}^{\infty} (j+1) \{ \phi_x(2j+2) - \phi_x(2j) \},$$

where

$$\phi_x(y) = \frac{\phi(y)}{2x\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\phi^2(x)t^2/8x^2}\, dt.$$

PROOF . Using well known properties of the modulus of continuity , we have

$$|M_{n,x}[f(t)] - f(x)| \leq \{1 + M_{n,x} [[\frac{|t-x|}{\phi(x)n^{-1/2}}]]\}\omega_f(\phi(x)n^{-1/2})$$

$$= C_n^*(x)\ \omega_f(\phi(x)n^{-1/2}),$$

where [.] stands for the integral part and $C_n^* = 1 + M_{n,x}[[\frac{|t-x|}{\phi(x)n^{-1/2}}]]$

Thus $C_n(x) \leq C_n^*(x)$. We next show that for our operators $C_n^*(x) = C_n(x)$.

Let ϵ be a positive number less than $\phi(x)n^{-1/2}$. Consider the function $f_{\epsilon,\ \phi(x)n^{-1/2}}(t)$ defined on $\mathbb{R}^+$ as follows

(i) $f_{\epsilon,\ \phi(x)n^{-1/2}}(x) = 0,$

(ii) $f_{\epsilon,\ \phi(x)n^{-1/2}}(t) = K,\ t \in [x+(K-1)\phi(x)n^{-1/2}+\epsilon,\ x+K\phi(x)n^{-1/2}]$

$(K = 1,2,\ldots),$

(iii) $f_{\epsilon,\ \phi(x)n^{-1/2}}(t)$ is linear on

$[x + (K-1)\ \phi(x)n^{-1/2},\ x + (K-1)\ \phi(x)n^{-1/2} + \epsilon]$

$(K = 1,2,\ldots)$, and

(iv) $f_{\epsilon,\ \phi(x)n^{-1/2}}(t)$ is symmetric about $t = x$.

It is easy to see that $\omega_{f_{\epsilon,\ \phi(x)n^{-1/2}}}(x,\ \phi(x)n^{-1/2}) = 1,$ and

$$\lim_{\epsilon \to 0} f_{\epsilon,\ \phi(x)n^{-1/2}}(t) = \begin{cases} 1 + \left[\dfrac{|t-x|}{\phi(x)n^{-1/2}}\right], & t \neq x \pm K\ \phi(x)n^{-1/2} \\ K, & t = x \pm K\ \phi(x)n^{-1/2} \end{cases}$$

$(K = 0,1,2,\ldots).$

Since the points of the form $x \pm Kn^{-1/2}\phi(x)$, $K = 0,1,2,\ldots$ constitute a set of measure zero and $M_{n,x}$ is an integral-type operator, using Lebesgue's dominated convergence theorem, we can write

$$(2.12)\qquad \lim_{\epsilon \to 0} M_{n,x}[f_{\epsilon,\ \phi(x)n^{-1/2}}(t)] = 1 + M_{n,x}\left[\left[\frac{|t-x|}{\phi(x)n^{-1/2}}\right]\right] = C_n^*(x)$$

According to the definition of $C_n(x)$,

$$|M_{n,x}[f_{\epsilon,\ \phi(x)n^{-1/2}}(t)]| \leq C_n(x)\ \omega_{f_{\epsilon,\ \phi(x)n^{-1/2}}}(\phi(x)n^{-1/2}) = C_n(x)$$

Hence, by (2.12),

$$C_n^*(x) = \lim_{\epsilon \to 0} M_{n,x}[f_{\epsilon,\ \phi(x)n^{-1/2}}(t)] \leq C_n(x).$$

It follows that $C_n^*(x) = C_n(x)$.

Now we determine the asymptotic value of

$$C_n(x) = M_{n,x}\left[1 + \left[\frac{|t-x|}{\phi(x)n^{-1/2}}\right]\right] = M_{n,x}[f_o], \text{ say.}$$

First, we observe that with $1/3 < \gamma < 1/2$,

$$(2.13)\qquad M_{n,x}[f_o] \cong M_{n,x}[f_o \chi_\gamma],$$

where χ_γ is the characteristic function of the interval $[x - \phi(x)n^{-\gamma},\ x + \phi(x)n^{-\gamma}]$. For, if m is a positive integer,

$$M_{n,x}[(1-\chi_\gamma)f_o] \leq M_{n,x}\left[\left(\frac{t-x}{\phi(x)n^{-\gamma}}\right)^{2m}\left(1 + \frac{|t-x|}{\phi(x)n^{-1/2}}\right)\right]$$

$$= O(n^{-m+2m\gamma}), \qquad (\text{by}(2.3))$$

(2.14)

$$= o(1), \qquad (\text{since } \gamma < 1/2)$$

Thus (2.13) is obvious from (2.14) and the fact that $M_{n,x}[f_o] \geq 1$.

In view of (2.13) and the result of (2.14), we can write

$$M_{n,x}[f_o] = \frac{n^n}{\Gamma(n)x^{n+1}} \left[\left(\int_0^x + \int_x^\infty\right) f_o(t)\, t^n e^{-nt/x}\, dt \right]$$

(2.15)
$$\simeq \sum_{j=-[n^{\{1/2\}-\gamma}]-1}^{0} (-j+1) \frac{n^n}{\Gamma(n)x^{n+1}} \int_{x+(j-1)\phi(x)n^{-1/2}}^{x+j\phi(x)n^{-1/2}} t^n e^{-nt/x}\, dt$$

$$+ \sum_{j=0}^{[n^{\{1/2\}-\gamma}]+1} (j+1) \frac{n^n}{\Gamma(n)x^{n+1}} \int_{x+j\phi(x)n^{-1/2}}^{x+(j+1)\phi(x)n^{-1/2}} t^n e^{-nt/x}\, dt$$

$$= \Sigma_1 + \Sigma_2, \text{ say.}$$

We consider Σ_1 and Σ_2 separately.

$$\Sigma_1 = \sum_{j=-[n^{\{1/2\}-\gamma}]-1}^{0} (-j+1) \frac{n^{n-\{1/2\}}\phi(x)}{\Gamma(n)\ 2x} \int_{2j-2}^{2j} \left(1+\frac{\phi(x)}{2x}sn^{-1/2}\right)^n e^{-n\left(1+\frac{\phi(x)}{2x}sn^{-1/2}\right)} ds$$

$$= \sum_{j=-[n^{\frac{1}{2}-\gamma}]-1}^{0} (-j+1) \frac{n^{n-\frac{1}{2}}\phi(x)}{\Gamma(n)\ 2x} \int_{2j-2}^{2j} \exp\left\{\frac{\phi(x)}{2x}sn^{1/2} - \frac{\phi^2(x)s^2}{8x^2} + o(1)\right\} e^{-n\left(1+\frac{\phi(x)}{2x}sn^{-\frac{1}{2}}\right)} ds$$

(o(1) holding uniformly in s and j)

$$\cong \sum_{j=-[n^{\{1/2\}-\gamma}]-1}^{0} (-j+1) \frac{\phi(x)}{2x\sqrt{2\pi}} \int_{2j-2}^{2j} e^{-\frac{\phi^2(x)s^2}{8x^2}} ds .$$

A similar analysis with Σ_2 yields

$$\Sigma_2 \cong \sum_{j=0}^{[n^{\{1/2\}-\gamma}]+1} (j+1) \frac{\phi(x)}{2x\sqrt{2\pi}} \int_{2j}^{2j+2} e^{-\frac{\phi^2(x)s^2}{8x^2}} ds$$

Notice that the above asymptotic values of Σ_1 and Σ_2 are equal. Therefore,

$$(2.16)\quad M_{n,x}[f_o] \cong 2 \sum_{j=0}^{[n^{\{1/2\}-\gamma}]+1} (j+1) \frac{\phi(x)}{2x} \frac{1}{\sqrt{2\pi}} \int_{2j}^{2j+2} e^{-\frac{\phi^2(x)s^2}{8x^2}} ds$$

$$\cong 2 \sum_{j=0}^{\infty} (j+1) \frac{\phi(x)}{2x} \frac{1}{\sqrt{2\pi}} \int_{2j}^{2j+2} e^{-\frac{\phi^2(x)s^2}{8x^2}} ds$$

$$= 2 \sum_{j=0}^{\infty} (j+1) \{ \phi_x(2j+2) - \phi_x(2j) \} .$$

This completes the proof.

As a consequence of Theorem 2.5.1, we have the following results :

<u>THEOREM 2.5.2</u> : If

$$(2.17)\quad B_n(x) = \inf \{ C \quad |M_{n,x}[f(t)] - f(x)| \leq C\, \omega_f(n^{-1/2}),$$

$$\text{for all continuous } f \in B^* \},\ (x > 0),$$

then

$$(2.18) \qquad B_n(x) \cong B_\infty(x) = 2 \sum_{j=0}^{\infty} (j+1)\{\phi_x^*(2j+2) - \phi_x^*(2j)\},$$

$$\text{where} \quad \phi_x^*(y) = \frac{1}{2x\sqrt{2\pi}} \int_{-\infty}^{y} e^{-s^2/8x^2}\, ds.$$

THEOREM 2.5.3 If

$$(2.19) \qquad D_n = \inf \{C \cdot |M_{n,x}[f(t)] - f(x)| \le C\, \omega_f(2xn^{-1/2})$$

$$\text{for all continuous } f \in B^*\}, \quad (x > 0),$$

then

$$(2.20) \qquad D_n \cong D_\infty = 2 \sum_{j=0}^{\infty} (j+1)\{\phi^{**}(2j+2) - \phi^{**}(2j)\}$$

$$= 1.045564\ldots,$$

$$\text{where} \quad \phi^{**}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-s^2/2}\, ds.$$

Notice that in the last theorem the constants D_n and D_∞ are independent of x and moreover that D_∞ is the same constant as occurs in the work of Esseen [18] for Bernstein polynomials.

CHAPTER III

DIRECT, INVERSE AND SATURATION THEOREMS

3 1 INTRODUCTION

Let L_n be a sequence of linear operators and let ψ_n be a sequence of positive real numbers which converges to zero as $n \to \infty$. We say that L_n is 'saturated' with order ψ_n in a certain norm, if $||L_n[f]-f|| = o(\psi_n)$ holds only for a 'trivial class' of functions and there exists atleast one function g such that $||L_n[g]-g|| = O(\psi_n)$. Then, the class of all functions g belonging to the domain of definition of L_n for which this relation holds, is called the 'saturation class' of the operators L_n

A 'direct theorem' determines the order of approximation in approximating f by $L_n[f]$ if f belongs to a certain class of functions On the other hand, an 'inverse theorem' decides the class to which f belongs if the order of approximation of f by $L_n[f]$ is given.

In this chapter we discuss three different approaches for proving direct and inverse theorems for the operators $M_{n,x}$. The first approach utilises Bernstein-type arguments and a commutativity property of the operators $M_{n,x}$. The theorems proved by this method are global, in nature. In the next section we show that certain iterates of $M_{n,x}$ converge to an

operator belonging to a semi-group of class (C_o) as has been the case with Bernstein polynomials and Szász operators [53]. With this we prove a direct and inverse theorem involving the limiting operators. Finally, in the last section we utilize certain results of May [52] to obtain local inverse and saturation theorems for linear combinations of $M_{n,x}$.

3.2 COMMUTATIVITY APPROACH

First we shall prove some preliminary lemmas

LEMMA 3.2.1. The existence of any one of $M_{n,x}[M_{m,s}f]$ and $M_{m,x}[M_{n,s}f]$ implies that of the other and moreover the two are equal.

The result easily follows from the Fubini theorem.

LEMMA 3.2.2 Let f be continuous on $[a,b]$. Then for all x and $t>0$ such that $x+t,\ x-t \in [a,b]$, there holds,

$$(3.1)\qquad \left|\frac{f(x+t)+f(x-t)}{2} - f(x)\right| \leq \frac{Mt^2}{2}$$

if, and only if $f'(x)$ exists and belongs to $Lip_M\ 1$

PROOF To prove the necessity part let

$$\lambda_t(x) = \frac{1}{t^2}\int_a^x \left\{\frac{f(\sigma+t)+f(\sigma-t)}{2} - f(\sigma)\right\} d\sigma$$

Then, $|\lambda_t(x) - \lambda_t(y)| \leq \frac{M}{2}|x-y|$. Hence $\lambda_t(x)$ is of bounded variation on $[a,b]$, the total variation bounded by a quantity not depending on t. Now, since $\{\lambda_{t_n}\}\ (t_n \to 0,\ n \to \infty)$

is a sequence of functions of uniformly bounded variation, Helley's selection theorem guarantees the existence of a subsequence $\{\lambda_{t_{n_k}}\}_{k=1}^{\infty}$ of $\{\lambda_{t_n}\}$ such that $\{\lambda_{t_{n_k}}\}$ converges pointwise to a function $\lambda(x)$ of bounded variation We consider the behaviour of

$$\int_a^b \phi(x) \frac{1}{t_{n_k}^2} \{ \frac{f(x+t_{n_k})+ f(x-t_{n_k})}{2} - f(x) \} dx \text{ as } k \to \infty,$$

where $\phi(x) \in C_0^{\infty}(a,b)$, the space of infinitely differentiable functions having a compact support in (a,b). We have

$$\lim_{k\to\infty} [\frac{1}{t_{n_k}^2} \int_a^b \phi(x) \{ \frac{f(x+t_{n_k}) + f(x-t_{n_k})}{2} - f(x) \} dx]$$

$$= \lim_{k\to\infty}[\frac{1}{t_{n_k}^2} \int_{a+t_{n_k}}^{b+t_{n_k}} \frac{\phi(x-t_{n_k})}{2} f(x)dx + \int_{a-t_{n_k}}^{b-t_{n_k}} \frac{\phi(x+t_{n_k})}{2} f(x)dx - \int_a^b \phi(x)f(x)dx]$$

(3.?)

$$= \lim_{k\to\infty} [\int_a^b \frac{\phi(x-t_{n_k}) + \phi(x+t_{n_k}) - 2\phi(x)}{2t_{n_k}^2} f(x)\, dx]$$

$$+ \lim_{k\to\infty} [\frac{1}{t_{n_k}^2} \int_b^{b+t_{n_k}} \frac{\phi(x-t_{n_k})}{2} f(x)dx - \int_{b-t_{n_k}}^{b} \frac{\phi(x+t_{n_k})}{2} f(x)dx -$$

$$\int_a^{a+t_{n_k}} \frac{\phi(x-t_{n_k})}{2} f(x)dx + \int_{a-t_{n_k}}^{a} \frac{\phi(x+t_{n_k})}{2} f(x)dx] .$$

Since $\phi(x) \in C_0^\infty(a,b)$, the last four integrals on the right hand side of (3.2) vanish Hence

$$\lim_{k\to\infty} \int_a^b \phi(x) \frac{1}{t_{n_k}^2} \left\{ \frac{f(x+t_{n_k}) + f(x-t_{n_k})}{2} - f(x) \right\} dx$$

(3.3) $$= \lim_{k\to\infty} \int_a^b \frac{\phi(x-t_{n_k}) + \phi(x+t_{n_k}) - 2\phi(x)}{2t_{n_k}^2} f(x)dx$$

$$= \frac{1}{2} \int_a^b \phi''(x)\, f(x)\, dx.$$

But, the left hand side of (3.3) is

(3.4) $$\lim_{k\to\infty} \int_a^b \phi(x)\, d\lambda_{t_{n_k}}(x) = \int_a^b \phi(x) d\lambda(x),$$

(by Helley's selection theorem).

Thus, (3.3) and (3.4) give

(3.5) $$\int_a^b \phi(x)\, d\lambda(x) = \frac{1}{2} \int_a^b \phi''(x)f(x)dx$$

Now, integrating by parts, we have

$$\int_a^b \phi(x)d\lambda(x) = \int_a^b \phi''(x) \int_a^x \lambda(t)dt\, dx.$$

Therefore, from (3.5)

$$\int_a^b \phi''(x) \left\{ \frac{1}{2} f(x) - \int_a^x \lambda(t)dt \right\} dx = 0, \quad \phi \in C_0^\infty(a,b).$$

It follows from [Lemma 1 1, 53] that

$\frac{1}{2} f(x) - \int_a^x \lambda(t)\, dt$ is a linear function on (a,b).

Thus $f'(x) = 2\lambda(x) +$ constant.

Since $\lambda(x) \in Lip_{M/2}\, 1$, $f'(x) \in Lip_M\, 1$.

This completes the proof of the necessity part.

Conversely, let $f'(x)$ exist and belong to $Lip_M\, 1$.

We have

$$f(x+t) = f(x) + t\, f'(x) + \int_x^{x+t} [f'(s)-f'(x)]\, ds, \text{ and}$$

$$f(x-t) = f(x) - t\, f'(x) - \int_{x-t}^{x} [f'(s)-f'(x)]\, ds.$$

Therefore,

$$|f(x+t) + f(x-t) - 2f(x)|$$

$$\leq \left| \int_x^{x+t} [f'(s)-f'(x)]\, ds \right| + \left| \int_{x-t}^{x} [f'(s)-f'(x)]\, ds \right|$$

$$\leq \int_x^{x+t} |f'(s)-f'(x)|\, ds + \int_{x-t}^{x} |f'(s)-f'(x)|\, ds$$

$$\leq M \int_x^{x+t} (s-x)\, ds - M \int_{x-t}^{x} (s-x)\, ds$$

$$= Mt^2,$$

which proves the lemma

<u>LEMMA 3.2.3</u> Let $f \in B^*$ and $|f(t)| \leq Mt^{\alpha}$, for some constants M and $\alpha > 0$. Then

$$(3.6) \qquad \frac{\partial^2}{\partial x^2} \{x\, M_{n,x}\, [f(t)]\} \leq \frac{C\, n\, M}{x^{1-\alpha}},$$

where C is a constant depending on α only.

PROOF. By the definition of $M_{n,x}$,

$$x\, M_{n,x}\,[f(t)] = \frac{n^n}{\Gamma(n)x^n} \int_0^\infty f(t)\,t^n\, e^{-nt/x}\,dt.$$

In view of the relation

$$\frac{\partial^2}{\partial x^2}\left(\frac{e^{-nt/x}}{x^n}\right) = \left\{-\frac{2n}{x^3}(t-x) - \frac{n}{x^2} + \frac{n^2}{x^4}(t-x)^2\right\} \frac{e^{-nt/x}}{x^n},$$

there follows,

$$\frac{\partial^2}{\partial x^2}\{x\, M_{n,x}\,[f(t)]\}$$

$$\leq M_{n,x}\left[\left\{\frac{2n}{x^2}\,|t-x| + \frac{n}{x} + \frac{n^2}{x^3}\,(t-x)^2\right\}|f(t)|\right]$$

$$\leq M\ \frac{n}{x}\, M_{n,x}\left[\frac{2}{x}\,|t-x|\,t^\alpha + t^\alpha + \frac{n}{x^2}\,(t-x)^2 t^\alpha\right]$$

$$= \frac{CnM}{x^{1-\alpha}},$$

by (1.6),(1.9),(2.3) and Schwarz's inequality.

This completes the proof

Now, we prove a direct and inverse theorem for the operators $M_{n,x}$.

THEOREM 3.2.1 Let $f \in B^*$. Then, for all sufficiently large n,

$$(3.7) \qquad |M_{n,x}\,[f(t)] - f(x)| \leq \frac{Mx}{2n},$$

if, and only if $\{xf(x)\}' \in \mathrm{Lip}_M 1$.

PROOF Let $\{xf(x)\}' \in \mathrm{Lip}_M 1$ Putting $xf(x) = F(x)$, we have

$$F(t) - F(x) = (t-x)F'(x) + \int_x^t [F'(s) - F'(x)]\,ds.$$

Hence

$|F(t) - F(x) - (t-x) F'(x)| \leq |\int_x^t [F'(s) - F'(x)] ds|$

Therefore,

$$\left|\frac{n^n}{\Gamma(n)x^n} \int_0^\infty F(t)t^{n-1} e^{-nt/x} dt - F(x) - \frac{n^n}{\Gamma(n)x^n}F'(x)\int_0^\infty (t-x)t^{n-1}e^{-nt/x}dt\right|$$

$$\leq \frac{M}{2} \frac{n^n}{\Gamma(n)x^n} \int_0^\infty (t-x)^2 t^{n-1} e^{-nt/x} dt.$$

Evaluating the two sides, we get

$|M_{n,x} [f(t)] - f(x)| \leq \frac{Mx}{2n}$, proving the first part of the theorem

Conversely, if for all sufficiently large n

$$|M_{n,s} [f(t)] -f(s)| \leq \frac{Ms}{2n},$$

using Lemma 3 2.1 and the evaluation (1.8) for $M_{m,x}[t]$, we have

$$|M_{n,x} [M_{m,s} [f(t)]] - M_{m,x} [f(s)] | \leq \frac{Mx}{2n}(1 + \frac{1}{m}),$$

for all m sufficiently large.

An application of Theorem 1.5 4 leads to the inequality

$$(3.8) \quad \left| \frac{\partial^2}{\partial x^2} \{ x M_{m,x} [f(t)] \} \right| \leq M(1+ \frac{1}{m})$$

If $0 < s < x$ and $g_m(x) = x M_{m,x} [f(t)]$, we have

$$|g_m(x+s) + g_m(x-s) - 2g_m(x)| = s^2 |g_m''(\xi)|$$

for some ξ such that $x-s < \xi < x+s$ Then, from (3.8),

$$(3.9) \quad |g_m(x+s) + g_m(x-s) - 2g_m(x)| \leq Ms^2(1+\frac{1}{m}).$$

Taking limit as $m \to \infty$, we get $|F(x+s)+F(x-s)-2F(x)| \leq Ms^2$, where $F(x) = xf(x)$.

This, however, by Lemma 3.2.2 implies that $F'(x) = \{xf(x)\}' \in \mathrm{Lip}_M 1$, proving the theorem.

DEFINITION 3.2.1 We say that a function $f \in Z_\alpha^*$, $0 < \alpha \leq 2$, iff

(3.10) $\quad |f(x+hx)+f(x-hx)-2f(x)| \leq M(hx)^\alpha, \; 0 < h \leq c < 1, \; x > 0,$

where c and M are some constants.

Now we give below the main theorem of this section.

THEOREM 3.2.2 Let $f \in B^*$ such that for some α satisfying $0 < \alpha \leq 2$, $|t\, f(t)| \leq Mt^\alpha$ where M is a constant. Then, the following are equivalent

(i) $\quad xf(x) \in Z_\alpha^*$,

(ii) For some constant $A > 0$ and all n sufficiently large,

(3.11) $\quad |M_{n,x}[f(t)] - f(x)| < Ax^{\alpha-1} n^{-\alpha/2}.$

PROOF The case $\alpha = 2$ (in fact in a much stronger form) has already been covered by theorem 3.2.1. Hence let $0<\alpha<2$, and assume (ii) to be true. Let h be a number satisfying $0 < h \leq c < 1$, where c is a constant. We can determine a positive integer 'm' such that

(3.12) $\quad 2^{m-1} \leq \dfrac{1}{h^2} < 2^m.$

Define

$$U_o(x) = U_o(f,x) = M_{1,x}[M_{2,s}[f(t)]], \text{ and}$$

(3.13)

$$U_n(x) = U_n(f,x) = M_{2n,x}[(M_{2n+1,s} - M_{2n-1,s})[f(t)]], \quad n = 1,2,\ldots$$

It is clear that

(3.14) $$\sum_{n=o}^{N} U_n(x) = M_{2N,x}[M_{2N+1,s}[f(t)]].$$ Also we have

$$|M_{2N,x}[M_{2N+1,s}[f(t)]] - M_{2N,x}[f(s)]|$$

$$= |M_{2N,x}[M_{2N+1,s}[f(t)] - f(s)]| \to 0 \text{ as } N \to \infty$$

from(3.11).

Therefore,

(3.15) $$\lim_{N \to \infty} M_{2N,x}[M_{2N+1,s}[f(t)]] = \lim_{N \to \infty} M_{2N,x}[f(s)] = f(x)$$

From (3.14) and (3.15),

$$f(x) = \sum_{n=o}^{\infty} U_n(x).$$

Taking $xf(x) = F(x)$ and $V_n(x) = x\, U_n(x)$, we have

(3.16) $$F(x) = \sum_{n=o}^{\infty} V_n(x).$$

From (3.16) we have

$$|F(x+hx)+F(x-hx)-2F(x)|$$

$$\leq \sum_{n=o}^{\infty} |V_n(x+hx)+V_n(x-hx)-2V_n(x)|$$

(3.17) $$\leq \sum_{n=o}^{m-1} |V_n(x+hx)+V_n(x-hx)-2V_n(x)|$$

Now

$$|V_n(x)| = \left| x M_{2^n,x}[(M_{2^{n+1},s} - M_{2^{n-1},s})[f(t)]] \right|$$

$$\leq x M_{2^n,x}[\,|M_{2^{n+1},s}[f(t)] - f(s)|\,]$$

$$+ x M_{2^n,x}[\,|f(s) - M_{2^{n-1},s}[f(t)]|\,]$$

$$\leq x M_{2^n,x}[A s^{\alpha-1} 2^{-(n+1)\alpha/2}]$$

$$+ x M_{2^n,x}[A s^{\alpha-1} 2^{-(n-1)\alpha/2}] \text{ (by 3.11)}$$

$$= x A 2^{-(n+1)\alpha/2} M_{2^n,x}[s^{\alpha-1}]$$

$$+ x A 2^{-(n-1)\alpha/2} M_{2^n,x}[s^{\alpha-1}]$$

(3.18)

$$\leq A 2^{-(n+1)\alpha/2} C_\alpha x^\alpha + A 2^{-(n-1)\alpha/2} C_\alpha x^\alpha, \ C_\alpha \text{ is a constant,}$$

(by evaluations of Chapter I)

$$= B x^\alpha 2^{-n\alpha/2}, \text{ where } B = A C_\alpha (2^{-\alpha/2} + 2^{\alpha/2}).$$

From (3.18) we have

$$\sum_{n=m}^{\infty} \{|V_n(x+hx)| + |V_n(x-hx)| + 2|V_n(x)|\}$$

$$\leq 4B \{x(1+\theta h)\}^\alpha \sum_{n=m}^{\infty} 2^{-n\alpha/2}, \text{(for some } \theta \text{ such that } -1 < \theta < 1)$$

$$= \frac{4Bx^\alpha (1+\theta h)^\alpha 2^{-m\alpha/2}}{1-2^{-\alpha/2}}$$

Thus

$$(3.19)\qquad \sum_{n=m}^{\infty}\{|V_n(x+hx)|+|V_n(x-hx)|+2|V_n(x)|\}\le K\,x^{\alpha}\,2^{-m\alpha/2},$$

where $K = 4B\,2^{\gamma}(1-2^{-\alpha/2})$

Now,

$$|(M_{2^{n+1},s} - M_{2^{n-1},s})[f(t)]|$$

$$= \left|(M_{2^{n+1},s} - M_{2^{n-1},s})[f(t)] - f(s) + f(s)\right|$$

$$\le |M_{2^{n+1},s}[f(t)] - f(s)| + \left|M_{2^{n-1},s}[f(t)] - f(s)\right|$$

$$\le As^{\alpha-1}\,2^{-(n+1)\alpha/2} + As^{\alpha-1}\,2^{-(n-1)\alpha/2},\text{ from the hypothesis}$$

$$\le A' s^{\alpha-1}\,2^{-n\alpha/2},\text{ for some constant } A'.$$

Therefore from Lemma 3.2.3,

$$\frac{\partial^2}{\partial x^2}\{x\,M_{2^n,x}[(M_{2^{n+1},s} - M_{2^{n-1},s})[f(t)]\}\le \frac{A'\,C\,2^{-n\alpha/2}\,2^n}{x^{2-\alpha}}$$

Hence

$$(3.20)\qquad |V_n''(x)| = \left|\frac{\partial^2}{\partial x^2}\{x\,M_{2^n,x}[(M_{2^{n+1},s} - M_{2^{n-1},s})[f(t)]]\}\right|$$

$$\le \frac{A'C\,2^{-n\alpha/2}2^n}{x^{2-\alpha}} = \frac{B'\,2^{n(1-\alpha/2)}}{x^{2-\alpha}},\text{ where } B' = A'C.$$

With $0 < h \le c < 1$, from (3.20),

$$|V_n(x+hx)+V_n(x-hx)-2V_n(x)\qquad \le h^2x^2\,V_n''(\xi),\ (\text{where } \xi = x(1+\theta_n' h),$$
$$-1<\theta_n'<1)$$

$$\leq \frac{h^2 x^2 B' 2^{n(1-\alpha/2)}}{x^{2-\alpha}\ (1+\theta'_n h)^{2-\alpha}}$$

$$= \frac{h^2 B'}{(1+\theta'_n h)^{2-\alpha}}\ x^{\alpha}\ 2^{n(1-\alpha/2)}$$

Therefore, we have

$$\sum_{n=0}^{m-1} |V_n(x+hx)+V_n(x-hx)-2V_n(x)|$$

$$\leq h^2 B'\ x^{\alpha} \sum_{n=0}^{m-1} \frac{2^{n(1-\alpha/2)}}{(1+\theta_n' h)^{2-\alpha}}$$

$$\leq \frac{h^2 B'}{(1+\theta'' h)^{2-\alpha}}\ x^{\alpha} \sum_{n=0}^{m-1} 2^{n(1-\alpha/2)}, \quad (\text{where } \theta'' = \min \theta'_n ,$$

(3.21) $$0 \leq n \leq m-1)$$

$$< \frac{h^2 B'}{(1+\theta'' h)^{2-\alpha}}\ x^{\alpha}\ \frac{2^{m(1-\alpha/2)}}{2^{1-\alpha/2} - 1}.$$

Thus, from (3.17),(3.19) and (3.21),

$$|F(x+hx)+F(x-hx)-2F(x)|$$

$$\leq K\ x^{\alpha}\ 2^{-m\alpha/2} + \frac{h^2 B'}{(1+\theta'' h)^{2-\alpha}}\ x^{\alpha}\ \frac{2^{m(1-\alpha/2)}}{2^{1-\alpha/2} - 1}$$

$$\leq K\ x^{\alpha}\ h^{\alpha} + \frac{h^2 B'}{(1+\theta'' h)^{2-\alpha}(2^{1-\alpha/2} - 1)}\ x^{\alpha}\ \left(\frac{2}{h^2}\right)^{1-\alpha/2}, \ \text{from}(3\ 12)$$

$$= \{K + \frac{B'\ 2^{1-\alpha/2}}{(1+\theta'' h)^{2-\alpha}\ (2^{1-\alpha/2} - 1)}\}\ (hx)^{\alpha}$$

$$\leq \; M(hx)^{\alpha},$$

since $0 < h \leq c < 1$, where $M = \{K + \dfrac{B' 2^{1-\alpha/2}}{(1-c)^{2-\alpha}(2^{1-\alpha/2}-1)}\}$.

This implies that $F(x) \in Z_{\alpha}^{*}$, i.e., $x\,f(x) \in Z_{\alpha}^{*}$, proving the implication (ii)$\Rightarrow$(i).

To prove the converse let $F(x) = xf(x) \in Z_{\alpha}^{*}$. Then

(3.22) $\qquad |F(x+hx)+F(x-hx)-2F(x)| \leq M(hx)^{\alpha}$. We have from the definition of $M_{n,x}[f]$,

$$M_{n,x}[f(t)] - f(x) = \frac{1}{x}\{\frac{n^n}{\Gamma(n)x^n}\int_0^{\infty} f(t)t^n e^{-nt/x}\,dt - F(x)\}$$

$$= \frac{1}{x}\{\frac{n^n}{\Gamma(n)x^n}\int_0^{\infty} F(t)t^{n-1} e^{-nt/x}\,dt - F(x)\}$$

(3.23)

$$= \frac{1}{x}\;\{\frac{n^n}{\Gamma(n)}\int_0^{\infty}[F(tx)-F(x)]\,t^{n-1}e^{-nt}\,dt\}.$$

Now, let $|t-1| < n^{-\gamma}$, where $1/3 < \gamma < 1/2$. Then $t = 1 + \theta n^{-\gamma}$, $-1 < \theta < 1$. Let

$$I_1 = \int_{1-n^{-\gamma}}^{1+n^{-\gamma}} [F(xt)-F(x)]\,t^{n-1}e^{-nt}\,dt$$

$$= \int_{-1}^{1} [F(x(1+\theta n^{-\gamma}))-F(x)]\,(1+\theta n^{-\gamma})^{n-1}\,e^{-n(1+\theta n^{-\gamma})}n^{-\gamma}\,d\theta.$$

But, $(1+\theta n^{-\gamma})^{n-1} = \exp(n-1)\{\theta n^{-\gamma} - \dfrac{\theta^2 n^{-2\gamma}}{2} + \dfrac{\theta^3 n^{-3\gamma}}{3} - \ldots\}$

$$= \exp\{\theta n^{1-\gamma} - \frac{\theta^2}{2}n^{1-2\gamma} + o(1)\},$$

where $o(1)$ term holds uniformly with respect to $1 < \theta < 1$.

Therefore,

$$I_1 = \int_{-1}^{1} [F(x(1+\theta n^{-\gamma}))-F(x)]\ e^{-n+o(1)} \exp\{-\frac{\theta^2}{2} n^{1-2\gamma}\} n^{-\gamma}\ d\theta$$

$$= n^{-\gamma}\ e^{-n+o(1)} \int_0^1 [F(x(1+\theta n^{-\gamma}))+F(x(1-\theta n^{-\gamma}))-2F(x)] \exp(-\frac{\theta^2}{2} n^{1-2\gamma}) d\theta .$$

Hence from (3.23),

$$\left| x\{ M_{n,x}[f(t)] - f(x)\} \right|$$

$$\leq n^{-\gamma} e^{-n+o(1)} \frac{n^n}{\Gamma(n)} \int_0^1 \left| F(x(1+\theta n^{-\gamma}))+F(x(1-\theta n^{-\gamma}))-2F(x) \right| \exp(-\frac{\theta^2}{2} n^{1-2}) d\theta$$

$$+ |I_2| \frac{n^n}{\Gamma(n)} , \text{ (where } I_2 = \int_0^\infty \{F(tx)-F(x)\}\ t^{n-1} e^{-nt} dt - I_1)$$

$$\leq n^{-\gamma} e^{-n+o(1)} \frac{n^n}{\Gamma(n)} \int_0^1 M(x\theta n^{-\gamma})^\alpha \exp\{-\frac{\theta^2}{2} n^{1-2\gamma}\} d\theta$$

(3.24)

$$+ |I_2| \frac{n^n}{\Gamma(n)} , \qquad \text{(from the hypothesis)}$$

$$= Mx^\alpha n^{-\gamma(1+\alpha)} e^{-n+o(1)} \frac{n^n}{\Gamma(n)} \int_0^1 \theta^\alpha \exp\{-\frac{\theta^2}{2} n^{1-2\gamma}\} d\theta$$

$$+ |I_2| \frac{n^n}{\Gamma(n)}$$

Now we shall consider each term on the right hand side of (3.24) separately. We have

$$\int_0^1 \theta^\alpha \exp\{-\frac{\theta^2}{2} n^{1-2\gamma}\} d\theta$$

$$\simeq \int_0^\infty \left(\frac{2s}{n^{1-2\gamma}}\right)^{\alpha/2} e^{-s} n^{-1/2+\gamma} \frac{ds}{\sqrt{s}} \quad (n\to\infty , \text{ putting } \frac{\theta^2}{2} n^{1-2\gamma} = s,$$

$$d\theta = \frac{ds}{\sqrt{s}\sqrt{n^{1-2\gamma}}})$$

$$= n^{\{-1/2\}+\gamma-\{\alpha/2\}+\alpha\gamma} \int_0^\infty s^{\{(\alpha+1)/2\}-1} e^{-s}\, ds$$

$$= \Gamma\left(\frac{\alpha+1}{2}\right) n^{-\{1/2\}+\gamma-\{\alpha/2\}+\alpha\gamma}.$$

Therefore, using Stirling's formula we can write

(3.25) $$M x^\alpha n^{-\gamma(1+\alpha)} e^{-n+o(1)} \frac{n^n}{\Gamma(n)} \int_0^1 \theta^\alpha \exp\{-\frac{\theta^2}{2} n^{1-2\gamma}\} d\theta \le A_1 x^\alpha n^{-\alpha/2},$$

for a suitable constant A_1.

Also

$$\frac{n^n}{\Gamma(n)} |I_2| \le \frac{n^n}{\Gamma(n)} \int_{|t-1| \ge n^{-\gamma}} |F(tx) - F(x)|\, t^{n-1} e^{-nt}\, dt$$

$$\le \frac{n^n}{\Gamma(n)} \int_{|t-1| \ge n^{-\gamma}} M x^\alpha (t^\alpha+1) t^{n-1} e^{-nt}\, dt$$

(from the hypothesis)

(3.26) $$= M x^\alpha \frac{n^n}{\Gamma(n)} \int_{|t-1| \ge n^{-\gamma}} (t^\alpha+1) t^{n-1} e^{-nt}\, dt$$

$= M x^\alpha\, O(n^{-\alpha/2})$ (by an analysis as done in the proof of Lemma 1.4.3)

$\le A_2 x^\alpha n^{-\alpha/2}$, say.

From (3.24)-(3.26) we have,

$$|x \{M_{n,x} [f(t)] - f(x)\}| \le (A_1+A_2) x^\alpha n^{-\alpha/2}$$

$= A x^\alpha n^{-\alpha/2}$, (where $A = A_1 + A_2$).

or $\qquad |M_{n,x}[f(t)] - f(x)| \le A\, x^{\alpha-1}\, n^{-\alpha/2}$

This proves the implication (i)$\Rightarrow$ (ii), and the proof of the theorem is complete.

REMARK 3.2.1: For functions defined on (o,∞), we say that $f \in Z_\alpha$, $0 < \alpha \le 2$, if for every $x > t > o$,

$$|f(x+t)+f(x-t)-2f(x)| \le M\, t^\alpha, \text{ where } M \text{ is a constant.}$$

It is obvious that $Z_\alpha \subseteq Z_\alpha^*$ If, however we consider the classes $\bar{Z}_\alpha^* = \{ f \in Z_\alpha^* \;\; |f(t)| \le A\, t^\alpha, \text{for some constant } A\}$ and

$$\bar{Z}_\alpha = \{ f \in Z_\alpha \;\; |f(t)| \le A\, t^\alpha, \text{ for some constant } A\},$$

then $\qquad \bar{Z}_\alpha^* \equiv \bar{Z}_\alpha.$

To prove this, in view of the inclusion $Z_\alpha \subseteq Z_\alpha^*$, it is sufficient to show that $\bar{Z}_\alpha^* \subseteq \bar{Z}_\alpha$. To show this let $f \in \bar{Z}_\alpha^*$. Then,

$$|f(x+hx)+f(x-hx)-2f(x)| \le M(hx)^\alpha, \; 0 < h \le c < 1, (c \text{ a constant}).$$

Putting $t = hx$, we have for $t \le cx$, $|f(x+t)+f(x-t)-2f(x)| \le M\, t^\alpha$

Now let $x \ge t \ge cx$, then using $|f(t)| \le A\, t^\alpha$,

$$\sup_{x \ge t \ge cx} \frac{|f(x+t)+f(x-t)-2f(x)|}{t^\alpha} \le \frac{A}{c^\alpha} \{2^\alpha + (1-c)^\alpha + 2\}.$$

Therefore, for all $t > 0$, taking $M^* = \max\{M, \frac{A}{c^\alpha}(2^\alpha + (1-c)^\alpha + 2)\}$,

$$|f(x+t)+f(x-t)-2f(x)| \le M^*\, t^\alpha,$$

showing that $f \in \bar{Z}_\alpha$.

In view of Remark 3.2.1, Theorem 3.2.2 can be re-stated as

THEOREM 3.2.3 Let $f \in B^*$ and let $0 < \alpha \leq 2$. If $|tf(t)| \leq Mt^{\alpha}$ for some constant M and $t > 0$ then the following statements are equivalent

(1) $\{x f(x)\} \in Z_{\alpha}$,

(2) For some constant $A > 0$ and all n sufficiently large

$$|M_{n,x} [f(t)] - f(x)| \leq A x^{\alpha-1} n^{-\alpha/2}$$

3.3 LIMITS OF CERTAIN ITERATES OF POST-WIDDER OPERATORS

The main object of this section is to show that the limiting operators corresponding to certain sequences of iterates of $M_{n,x}$ form a semigroup of class (C_0). We shall also utilise this result in proving a direct and inverse theorem which involves the limiting operator. Also, by this approach we characterise the convexity of certain functions f via an inequality involving $M_{n,x}[f]$.

We start with some definitions

DEFINITION 3.3.1 Let X be a Banach space and $\{T_t, t \geq 0\}$ a one parameter family of bounded linear operators mapping X into itself. $\{T_t, t \geq 0\}$ is said to be a semigroup of class (C_0) on X provided that

(3.27)

(i) $T_{t+s} = T_t \cdot T_s$, $t, s \geq 0$ and,

(ii) $\lim_{t \to 0^+} T_t x = x$ for all $x \in X$.

The infinitesimal generator of the semigroup is defined as

$$Ax = \lim_{t \to 0^+} \frac{T_t x - x}{t}, \quad \text{whenever this limit exists}$$

The case $r = 1$ of the following result of Micchelli [53, Theorem 1.2] will be required in the sequel

THEOREM I Let $\{T_t \quad t \geq 0\}$ be a semigroup of class (C_0) on a Banach space X and let r be a non-negative integer, then

$$\lim_{t \to 0^+} < F, \frac{(T_t - I)^r x}{t^r} > = < (A^*)^r F, x >$$

for $F \in D((A^*)^r)$ and any $x \in X$, where $< , >$ denotes the pairing between the dual space X^* and X, i.e., $< F, x > = F(x)$, $F \in X^*$, $x \in X$ and A^* denotes the adjoint of A.

DEFINITION 3.3.2 For $\alpha, \beta > 0$, we define

$$C_{\alpha,\beta} = \{f \quad f \text{ is continuous on } (0,\infty) \text{ and}$$

$$\lim_{x \to \infty} \frac{f(x)}{1+x^\beta} = 0 \text{ and } \lim_{x \to 0} x^\alpha f(x) = 0\}.$$

Each $C_{\alpha,\beta}$ is normed with the norm $\|\ \|_{\alpha,\beta}$ defined by

$$\|f\|_{\alpha,\beta} = \max_{x \in (0,\infty)} \frac{x^\alpha f(x)}{1+x^{\alpha+\beta}}$$

It is easy to see that $\|.\|_{\alpha,\beta}$ is a norm. In order to show that each $C_{\alpha,\beta}$ is a Banach space we observe that if $\{f_n\}$ is a Cauchy sequence in $C_{\alpha,\beta}$ converging to a function f in $\|.\|_{\alpha,\beta}$, then f is a continuous function on $(0,\infty)$. It remains to show that $\lim_{x \to \infty} \frac{f(x)}{1+x^\beta} = 0$ and $\lim_{x \to 0} x^\alpha f(x) = 0$.

However, these follow by taking limits as $x \to \infty$ and $x \to o$, respectively, in

$$\frac{x^{\alpha}}{1+x^{\alpha+\beta}} \left| f_n(x) - f(x) \right| < \epsilon, \; n > N(\epsilon),$$

where ϵ is any preassigned arbitrary positive number.

DEFINITION 3.4.3 . The iterates of $M_{n,x}[f]$ are defined as follows

We put $M^{o}_{n,x} = I$ and

$$M^{r}_{n,x} = M_{n,x}(M^{r-1}_{n,x}), \; r \geq 1.$$

Here, $f \in D(M^{r}_{n,x})$ if, and only if $f \in D(M^{r-1}_{n,x})$ and $M^{r-1}_{n,x} \in D(M_{n,x})$. We notice that for each fixed pair (α, β) there holds $C_{\alpha,\beta} \subseteq D(M^{r}_{n,x})$ for all r, $x > o$ and for all n sufficiently large Moreover $M^{r}_{n,x} \cdot C_{\alpha,\beta} \to C_{\alpha,\beta}$.

Let Y be the set obtained by one point compactification of (o,∞) and let $C(Y)$, as usual, denote the space of all continuous functions on Y.

LEMMA 3.3.1 . The set $\{e^{-\lambda x}, \lambda > o\}$ spans a dense subspace in each $C_{\alpha,\beta}$.

PROOF : Since $\left\{ \frac{x^{\alpha} e^{-\lambda x}}{1 + x^{\beta+\alpha}} \; ; \lambda > o \right\}$ is a family of functions belonging to $C(Y)$ and it separates points of Y, by the Stone-Weierstrass theorem the algebra S_1 generated by $\left\{1, \frac{x^{\alpha} e^{-\lambda x}}{1 + x^{\beta+\alpha}}, \lambda > o \right\}$ is dense in $C(Y)$. Since each function in $\left\{\frac{x^{\alpha} f(x)}{1 + x^{\beta+\alpha}}, f \in C_{\alpha,\beta} \right\}$ can be identified with a

function $\phi(x)$ belonging to $C(Y)$, given a function $f \in C_{\alpha,\beta}$ for every $\epsilon > o$ there exists $s_1 \in \left[\frac{x^{\alpha} e^{-\lambda x}}{1 + x^{\beta+\alpha}}, \lambda > o\right]$ such that

$$(3.28) \qquad \left\| \frac{x^{\alpha} f(x)}{1 + x^{\beta+\alpha}} - c_1 - s_1 \right\| < \epsilon/4,$$

where c_1 is a constant and $||\cdot||$ denotes the max-norm in $C(Y)$. Taking limit as $x \to \infty$ in (3.28), we get $c_1 < \epsilon/4$. Therefore (3.28) gives

$$(3.29) \qquad \left\| \frac{x^{\alpha} f(x)}{1 + x^{\beta+\alpha}} - s_1 \right\| < \epsilon/2.$$

Let s_o be the term consisting of first powers of $\left(\frac{x^{\alpha} e^{-\lambda x}}{1 + x^{\beta+\alpha}}\right)$'s in S_1 and let $s' = s_1 - s_o$. Then

$$\frac{1 + x^{\beta+\alpha}}{x^{\alpha}} s' \in C(Y^*),$$

where Y^* is the set obtained by one point compactification of $[0,\infty)$

Further, since $\{ e^{-\lambda x}, \lambda > o \}$ is [redacted] a separating family in $C(Y^*)$, the algebra S_2 generated by $\{ 1, e^{-\lambda x}, \lambda > o \}$ is [redacted] dense in $C(Y^*)$. Therefore, for every $\epsilon > o$ there exists $s_2 \in S_2$ such that

$$\left\| \frac{1 + x^{\beta+\alpha}}{x^{\alpha}} s' - s_2 \right\| < \epsilon/4, \quad \text{or}$$

$$(3.30) \qquad \left\| \frac{1 + x^{\beta+\alpha}}{x^{\alpha}} s_1 - c_2 - s \right\| < \epsilon/4,$$

where c_2 is a constant and $s \in [e^{-\lambda x}, \lambda > o]$.

Taking limit as $x \to \infty$ in (3.30), we get

$$c_2 < \epsilon/4.$$

Therefore,

$$(3.31) \qquad \left\| \frac{1 + x^{\beta+\alpha}}{x^{\alpha}} s_1 - s \right\| < \epsilon/2,$$

for some $s \in [e^{-\lambda x}, \lambda > 0]$. Thus,

$$\left\| s_1 - \frac{x^{\alpha}}{1 + x^{\beta+\alpha}} s \right\| \leq \left\| \frac{1+x^{\beta+\alpha}}{x^{\alpha}} s_1 - s \right\| \left\| \frac{x^{\alpha}}{1 + x^{\beta+\alpha}} \right\|$$

$$(3.32) \qquad < \left\| \frac{x^{\alpha}}{1 + x^{\beta+\alpha}} \right\| \epsilon/2 < \epsilon/2$$

Now, (3.29) and (3.32) together imply

$$\left\| \frac{x^{\alpha} f(x)}{1 + x^{\beta+\alpha}} - \frac{x^{\alpha}}{1 + x^{\beta+\alpha}} s \right\| < \epsilon,$$

i.e., $\| f(x) - s \|_{\alpha, \beta} < \epsilon$, for some $s \in [e^{-\lambda x}, \lambda > 0]$
This completes the proof of the lemma.

Next, we shall discuss, in brief, a few results of Micchelli[53] which we shall be using later in this section. Denote by C_α the space defined by $C_\alpha = \{f \ f$ continuous on $[0, \infty)$ and $\lim_{x \to \infty} \frac{f(x)}{1 + x^{\alpha}} = 0\}$. Each C_α is normed by $\|f\|_\alpha = \max_{0 \leq \lambda < \infty} \frac{|f(x)|}{1+x^{\alpha}}$. With this norm each C_α is a Banach space. We notice that C_α is contained in $C_{\alpha, \beta}$ for all $\beta > 0$. Further, since $\{e^{-\lambda x}, \lambda > 0\} \subseteq C_\alpha$, Lemma 3.3.1 implies that C_α is dense in $C_{\alpha, \beta}$.

Micchelli considers the following sequence of linear positive operators introduced by Karlin [34] Let $\{X_n , n \geq 0\}$ be a sequence of non-negative random variables. Suppose that the distribution function of X_n is F_n and that $E(X_n^r) = \int_0^\infty t^r \, dF_n(t) < \infty$ for all $n, r \geq 0$. Define a sequence of operators as follows .

$$U_n(f;x) = \int_0^\infty f\left(\frac{xt}{\mu_n}\right) \, dF_n(t), \quad 0 \leq x < \infty$$

where $\mu_n = E(X_n) = \int_0^\infty t \, dF_n(t) > 0$. Then U_n maps C_α into itself.

If, in addition, the sequence $\{X_n, n \geq 0\}$ satisfies the following conditions :

(i) $\lim_{n \to \infty} E(X_n) = +\infty$

(ii) $\lim_{n \to \infty} n \, E\left[\left(\frac{X_n}{E(X_n)} - 1\right)^2\right] = \sigma^2 > 0$

(iii) $\lim_{n \to \infty} n \, E\left[\left(\frac{X_n}{E(X_n)} - 1\right)^{2k}\right] = 0$ for $k = 2, 3, \ldots,$

then there holds [53] the following

<u>THEOREM II</u> : Let $f \in C_\alpha$ and $\mu_n = E(X_n)$. Then,

$$\lim_{n \to \infty} U_n^{[\mu_n t]}(f,x) = \mathcal{U}_t(f,x)$$

$$= \frac{1}{\sqrt{2\pi \sigma^2 t}} \int_{-\infty}^{\infty} f(xe^y) \exp\left[-\frac{1}{2\sigma^2 t}\left(y + \frac{t}{2}\sigma^2\right)^2\right] dy, \quad t > 0,$$

where $\mathcal{U}_t$ is a semigroup of class (C_0) on each C_α.

and its iterates are well defined if $n > \alpha$ and that

$M_{n,x}\,[f(s)] = \frac{1}{x}\, S_n^1\,(sf(s),x)$ It is clear from this that for each $r = 1,2,3,$ there holds

$$M_{n,x}^{r}\,[f(s)] = \frac{1}{x}\, S_n^{1^r}\,(sf(s),x). \tag{3.34}$$

Therefore, from Lemma 3.3.2 and (3 34) we have, for all $f \in C_\alpha$,

$$\lim_{n\to\infty} M_{n,x}^{[nt]}\,[f(s)] = \lim_{n\to\infty} \frac{1}{x}\, S_n^{1^{[nt]}}\,(sf(s),x)$$

$$= \frac{1}{x}\,\tau_t(sf(s),x). \tag{3.35}$$

We now show that for each fixed t there holds

$$\|M_{n,x}^{[nt]}\|_{\alpha,\beta} < K, \tag{3.36}$$

where K is a constant not depending on n.

For this, if $f \in C_{\alpha,\beta}$, we have

$$|f(x)| \le (x^{-\alpha} + x^{\beta})\,\|f\|_{\alpha,\beta}\,,\quad x > 0.$$

Operating over this inequality by $M_{n,x}^{[nt]}$, for some constant γ, we get

$$\frac{|M_{n,x}^{[nt]}[f(s)]|}{\|f\|_{\alpha,\beta}} \le \gamma\left(x^{-\alpha}\, e^{\frac{t\alpha(\alpha+1)}{2}} + x^{\beta}\, e^{\frac{t\beta(\beta-1)}{2}}\right), \tag{3.37}$$

(since $M_{n,x}^{[nt]}[s^k] \to e^{\frac{tk(k-1)}{2}}\, x^k$ as $n \to \infty$).

Therefore, from (3.37),

$$\|f\|_{\alpha,\beta}^{-1}\left|\frac{x^{\alpha}}{1+x^{\beta+\alpha}}\, M_{n,x}^{[nt]}[f(s)]\right| \le \gamma\left(e^{\frac{t\beta(\beta-1)}{2}} + \frac{e^{\frac{t\alpha(\alpha+1)}{2}} - e^{\frac{t\beta(\beta-1)}{2}}}{1+x^{\beta+\alpha}}\right)$$

Hence,

$$\frac{\|M_{n,x}^{[nt]}[f(s)]\|_{\alpha,\beta}}{\|f\|_{\alpha,\beta}} \le \gamma \max_{x\in(0,\infty)}\left(e^{\frac{t\beta(\beta-1)}{2}} + \frac{e^{\frac{t\alpha(\alpha+1)}{2}} - e^{\frac{t\beta(\beta-1)}{2}}}{1+x^{\beta+\alpha}}\right)$$

$$= \gamma \max\left(e^{\frac{t\beta(\beta-1)}{2}},\ e^{\frac{t\alpha(\alpha+1)}{2}}\right)$$

Takin supremum over $f \in C_{\alpha,\beta}$ we get (3.36)
Now the equations (3.35)-(3.36) and Lemmas 3.3.1 and 3.3.2 lead to the following important conclusions

(I) For all $f \in C_{\alpha,\beta}$,

(3.38) $\lim_{n\to\infty} M_{n,x}^{[nt]}[f(s)] = T_t(f,x)$, say, exists and

(II) $\{T_t, t \geq o\}$ is a semigroup of operators of class (C_o) on $C_{\alpha,\beta}$ for all $\alpha, \beta \geq o$

Summarizing, we have obtained the following

<u>THEOREM 3.3.1</u> . There exists a semigroup $\{T_t, t \geq o\}$ of class (C_o) on $C_{\alpha,\beta}$ such that

$$\lim_{n\to\infty} M_{n,x}^{[nt]}[f(s)] = T_t(f,x) \text{ for all } f \in C_{\alpha,\beta}$$

Further utilizing the relationship $T_t(f(s),x) = \frac{1}{x}\tau_t(sf(s),x)$, the infinitesimal generator $A(f,x)$ of $\{T_t, t \geq o\}$ has the form

(3.39) $A(f,x) = \frac{x}{2}(xf(x))''$, $x > o$ and moreover that if $\phi \in C_o^{\infty}(o,\infty)$ and $F_\phi(g) = \int_o^\infty \phi(x)\, g(x)\, dx, (g \in C_{\alpha,\beta})$ then $F_\phi \in D(A^*)$ and $A^*F_\phi = F_{A^*\phi}$ where A^* denotes the adjoint of A and $(A^*\phi)x = \frac{x}{2}(x\phi(\ ))''$.

Now we give below a direct and inverse theorem involving T_t.

<u>THEOREM 3.3.2</u> . Let $f \in \bigcup_{\alpha,\beta>o} C_{\alpha,\beta}$, then the following statements are equivalent.

(i) $(xf(x))'$ exists and belongs to $Lip_M 1$,

(ii) $|M_{n,x}[f(t)] - f(x)| \leq \frac{Mx}{2n}$ for all n sufficiently large and $x \in (o,\infty)$,

(iii) $|T_t(f,x) - f(x)| \leq \frac{Mx}{2}(e^t - 1)$ for all $t \geq o$ and $x \in (o,\infty)$

Consequently, if $|M_{n,x}[f(t)] - f(x)| \leq \epsilon_n x/n$, where $\lim_{n\to\infty} \epsilon_n = 0$, then $f(x)$ is a linear combination of 1 and $1/x$

PROOF The direct implication (i)$\Rightarrow$(ii) has already been proved in Theorem 3.2.1. Assume (ii) to be true. Then operating the inequality in (ii) by $M_{n,x}$, we get

$$|M^2_{n,x}[f(t)] - M_{n,x}[f(t)]| \leq \frac{M\,x}{2n}(1+\frac{1}{n}).$$

Reoperating this inequality by $M_{n,x}$, we have

$$|M^3_{n,x}[f(t)] - M^2_{n,x}[f(t)]| \leq \frac{M\,x}{2n}(1+\frac{1}{n})^2.$$

Continuing this operation $[nt]-1$ times, we get

$$|M^{[nt]}_{n,x}[f(s)] - M^{[nt]-1}_{n,x}[f(s)]| \leq \frac{M\,x}{2n}(1+\frac{1}{n})^{[nt]-1}$$

Therefore, adding all these inequalities, we get

$$(3.40) \qquad |M^{[nt]}_{n,x}[f(s)] - f(x)| \leq \frac{M\,x}{2n}\left\{\frac{(1+1/n)^{[nt]}-1}{1/n}\right\}.$$

Taking limit as $n\to\infty$ in (3.40) and using Theorem 3.3 1, we have

$$|T_t(f,x) - f(x)| \leq \frac{M\,x}{2n}(e^t - 1)$$

This proves the implication (ii)$\Rightarrow$(iii). Now, assume (iii) to be true. We verify (i) by showing that it is true for each closed interval $[a,b] \subset (0,\infty)$ Let $\phi \in C_0^\infty(a,b)$ and let

$$\lambda_t(x) = \frac{1}{(e^t-1)}\int_a^x \frac{T_t(f,\sigma) - f(\sigma)}{1/2\,\sigma}\,d\sigma \;.$$

Clearly, if (iii) holds then $\lambda_t \in \mathrm{Lip}_M 1$. Since $\phi \in C_0^\infty(a,b)$, we have

$$\lim_{t\to 0^+}\int_a^b \phi(x)\,\frac{T_t(f,x) - f(x)}{(e^t-1)}\,dx$$

$$(3.41) \qquad = \lim_{t\to 0^+}\int_a^b \phi(x)\,\frac{T_t(f,x)-f(x)}{t}\,\frac{t}{(e^t-1)}\,dx$$

$$= \lim_{t\to 0^+}\int_a^b \phi(x)\,\frac{T_t(f,x) - f(x)}{t}\,dx$$

$$= \int^b x\,f(x)\,\left(\frac{x\,\phi(x)}{}\right)''\,dx \text{ (by Theorems I and 3.3)}$$

Putting $\psi(x) = \frac{1}{2} x\, \phi(x)$ in (3.41), we have

$$(3.42) \qquad \lim_{t \to 0^+} \int_a^b \frac{\psi(x)}{(1/2)x} \frac{T_t(f,x) - f(x)}{(e^t - 1)}\, dx = \int_a^b \psi''(x) x f(x) dx.$$

But, the left hand side of (3.42), by Helley's selection theorem, is $\int_a^b \psi(x)\, d\lambda(x)$, for some function $\lambda(x) \in \mathrm{Lip}_M 1$ on (a,b). Therefore, for all $\psi \in C_0^\infty(a,b)$,

$$\int_a^b \psi(x)\, d\lambda(x) = \int_a^b \psi''(x)\, x\, f(x)\, dx,$$

i.e.,
$$\int_a^b \left(x\, f(x) - \int_a^x \lambda(t)\, dt\right) \psi''(x)\, dx = 0.$$

Hence from [53, Lemma 1.1] we conclude that

$$x\, f(x) - \int_a^x \lambda(t)\, dt + \alpha_1 + \beta_1 x,$$

where $x \in [a,b]$ and α_1, β_1 are constants. Since $\lambda(x) \in \mathrm{Lip}_M 1$, we see that $(x\, f(x))'$ exists and belongs to $\mathrm{Lip}_M 1$. This establishes the implication (iii)$\Rightarrow$(i), proving the theorem.

<u>THEOREM 3.3.3</u>. If $f \in \bigcup_{\alpha, \beta > 0} C_{\alpha,\beta}$ then $x\, f(x)$ is convex on $(0,\infty)$ if, and only if $M_{n,x}[f(s)] \geq f(s)$, for all $x \in (0,\infty)$ and n sufficiently large.

<u>PROOF</u> The operator $S_n^1(f,x)$ is a linear positive operator preserving linear functions Therefore, if $x\, f(x)$ is convex then $S_n^1(tf(t), x) \geq x f(x)$, i.e., $\frac{1}{x} S_n^1(tf(t),x) \geq f(x)$ and so from (3.34), $M_{n,x}[f(s)] \geq f(x)$

To prove the converse, let $M_{n,x}[f(s)] \geq f(x)$ for some $f \in \bigcup_{\alpha, \beta \geq 0} C_{\alpha,\beta}$. Operating this inequality by $M_{n,x}$ successively $[nt] - 1$ times, we get

$$M_{n,x}^{[nt]}[f(s)] \geq M_{n,x}^{[nt]-1}[f(s)] \geq \qquad \geq f(x).$$

Taking limit as $n \to \infty$ in this, from Theorem 3.3.1, we get

$$T_t(f,x) \geq f(x).$$

Now, let ϕ be a non-negative function in $C_0^\infty(0,\infty)$ Then, as in (3.41), we have

(3.43) $$0 \leq \lim_{t\to 0^+} \int_0^\infty \phi(x) \frac{T_t(f,x) - f(x)}{t} dx = \int_0^\infty (\tfrac{1}{2}x\phi(x))'' x f(x) dx$$

Since $\psi(x) = (1/2)x\,\phi(x)$ is a one-one-onto transformation on the set of all non-negative functions in $C_0^\infty(0,\infty)$, as in (3.43), we obtain

(3.44) $$\int_0^\infty \psi''(x)\, x\, f(x)\, dx \geq 0$$ for all non-negative functions $\psi \in C_0^\infty(0,\infty)$. Thus, from [53, Lemma 1.2] we conclude that $x\, f(x)$ is a convex function. This completes the proof of the theorem.

3.4 INVERSE AND SATURATION THEOREMS FOR LINEAR COMBINATIONS

May [52] considered a class of so called exponential-type operators defined by

$$S_\lambda(f,x) = \int_A^B w(\lambda,x,t)\, f(t)dt,$$

where the kernel $w(\lambda,x,t)$ is non-negative and satisfies,

$$\int_A^B w(\lambda,x,t)\, dt = 1 \text{ and}$$

$$\frac{\partial w}{\partial x} - \frac{\lambda}{p(x)} w(\lambda,x,t)(t-x),$$

where $p(x) > 0$ on (A,B), is a polynomial of degree two atmost. These operators are called regular if further,

$$\int_A^B w(\lambda,x,t)dt = a(\lambda)$$

where $a(\lambda)$ is a rational function of λ and $a(\lambda) \to 1$ as $\lambda \to \infty$

May considers the following linear combination of these operators.

Let $d_0, d_1, \ldots, d_k$ be k+1 arbitrary but fixed distinct positive integers. The operator $S_\lambda(f,k,t) \equiv (S_\lambda(f,k,t,d_0,d_1 \ldots d_k))$ is a linear combination of $S_{d_j\lambda}(f,t)$, defined by

$$\omega_\lambda(f,k,t) = \sum_{j=0}^{k} C(j,k)\, S_{d_j\lambda}(f,t),$$

where $C(j,k) = \prod_{\substack{i=0\\ i\neq j}}^{k} \frac{d_j}{d_j - d_i}$, $k \neq o$ and $C(o,o) = 1$

It may be noted that the linear combinations defined in (1.31) reduce to the above combinations on taking $\alpha_j = d_j$ an integer, $j = o,1,\ldots,k$

Let $\Delta_x^k f(t) = \sum_{\nu=o}^{k} (-1)^{k-\nu} \binom{k}{\nu} f(t+\nu x)$ and let $\omega_k(f,h,a,b) = \sup\{|\Delta_x^k f(t)|, |x| \leq h, t, t+kx \in [a,b]\}$ be the kth modulus of continuity of the function f. Then the class Liz (α,k,a,b) is the class of functions for which $\omega_{2k}(f,h,a,b) \leq M h^\alpha$. When $k = 1$, Liz $(\alpha,1)$ reduces to the well-known Zygmund class $Z_\alpha (\equiv Lip^* \alpha)$

Let $\psi \in C(A,B)$ be a positive growth test function satisfying $S_\lambda(\psi^2,x) < \infty$ uniformly on compact subsets of (A,B). Let $C_\psi(A,B) = \{f \in C(A,B) |\ |f(x)| \leq M\psi(x)$ for some $M > o\}$, endowed with the norm $||f||_\psi = \sup_{x\in(A,B)} |f(x)|\,\psi^{-1}(x)$. Also let $f \in C_\psi(A,B)$ and $A < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < B$ For the operators $S_\lambda(f,k,x)$ (without necessarily assuming the regularity condition) May obtained the following inverse theorem.

<u>THEOREM I</u> . Let $o < \alpha < 2$. Then in the following, the implications (1)$\Rightarrow$(2)$\Leftrightarrow$(3)$\Rightarrow$(4) hold .

(1) $||S_{\lambda_n}(f,k,x) - f(x)||_{C[a_1,b_1]} = O(\lambda_n^{-\alpha(k+1)/2}), (\lambda_{n+1}/\lambda_n \leq c$ for some $c > o)$,

(2) $f \in$ Liz $(\alpha, k+1, a_2, b_2)$,

(3) (i) For $m < \alpha(k+1) < m+1$, $m = 0,1,2,\ldots,2k$, $f^{(m)}$ exists and $f^{(m)} \in \mathrm{Lip}\,(\alpha(k+1) - m,\ a_2, b_2)$,

(ii) For $\alpha(k+1) = m+1$, $m = 0,1,2,\ \ ,2k$, $f^{(m)}$ exists and $f^{(m)} \in Z_1\,(a_2,b_2)$,

(4) $$||S_\lambda(f,k,x)-f(x)||_{C[a_3,b_3]} = O(\lambda^{-\alpha(k+1)/2}).$$

In addition assuming the regularity condition he obtained the following saturation theorem in the case $\alpha = 2$.

<u>THEOREM II</u> : In the following the implications (1)$\Rightarrow$(2)$\Rightarrow$(3) and (4)$\Rightarrow$(5)$\Rightarrow$(6) hold.

(1) $$\lambda_n^{k+1}||S_{\lambda_n}(f,k,x)-f(x)||_{C[a_1,b_1]} = O(1),\ (\lambda_{n+1}/\lambda_n \leq c),$$

(2) $$f^{(2k+1)} \in A.C.\,[a_2,b_2] \text{ and } f^{(2k+2)} \in L_\infty[a_2,b_2],$$

(3) $$\lambda^{k+1}\,||\,S_\lambda(f,k,x) - f(x)\,||_{C[a_3,b_3]} = O(1),$$

(4) $$\lambda_n^{k+1}\,||\,S_{\lambda_n}(f,k,x) - f(x)\,||_{C[a_1,b_1]} = o(1),(\lambda_{n+1}/\lambda_n \leq c),$$

(5) $f \in C^{2k+2}\,[a_2,b_2]$ and

$$\sum_{i=k+1}^{2k+2} Q(i,k,x)\,f^{(i)}(x) = 0,\ x \in [a_2,b_2],$$

where $Q(i,k,x)$ are certain polynomials in x depending on k,

(6) $$\lambda^{k+1}\,||S_\lambda(f,k,x) - f(x)||_{C[a_3,b_3]} = o(1).$$

The operators $S_n^1(f,x) = \frac{1}{\Gamma(n)}\left(\frac{n}{x}\right)^n \int_0^\infty f(t)\,t^{n-1}\,e^{-nt/x}\,dt$, which have also been termed Post-Widder operators by May, are also regular and of exponential type. Thereby the above theorems hold for these operators. The Post-Widder operators $M_{n,x}$, however, are not exponential type operators and the above results are not directly applicable as such. Nevertheless, as we show in the sequel, a simple transformation connects the operators $M_{n,x}$ with the operators S_n^1. Using this, we shall deduce from

the above theorems the corresponding slightly more general inverse and saturation theorems for linear combinations of the operators $M_{n,x}$. Considering the linear combination as defined in Chapter I, we notice that

$$(3.45)\qquad \begin{aligned} M_{n,x}[f(t)] &= \frac{1}{x} S_n^1(tf(t),x) \text{ and} \\ M_{n,x}^{[m]}[f(t)] &= \frac{1}{x} S_n^{1[m]}(tf(t),x) \end{aligned}$$

We also remark that Theorems I and II hold if we replace $S_\lambda(f,k,x)$ by our combination $S_n^{1[k+1]}(f,x)$ with α_i's and n not necessarily integers. This is atonce evident while going through the proofs of May [52].

We have the following inverse theorem for $M_{n,x}^{[k+1]}$, where we do not assume the continuity of f on $\mathbb{R}^+$.

<u>THEOREM 3.4.1</u> : Let $f \in B^*$, $o < \alpha < 2$ and $o < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < \infty$. Then, in the following, the implications (1)$\Rightarrow$(2)$\Leftrightarrow$(3)$\Rightarrow$(4) hold.

(1) $\displaystyle\sup_{x \in [a_1,b_1]} |M_{n_p,x}^{[k+1]}[f(t)] - f(x)| = O(n_p^{-\alpha(k+1)/2})$

$\left(\dfrac{n_{p+1}}{n_p} \le c \text{ for some constant } c > o\right)$,

(2) $f(x) \in \mathrm{Liz}\,(\alpha, k+1, a_2, b_2)$,

(3) (i) For $m < \alpha(k+1) < m+1$, $m = o,1,2, \quad ,2k+1$,

$f^{(m)}$ exists and belongs to $\mathrm{Lip}\,(\alpha(k+1)-m\,;\, a_2,b_2)$,

(ii) For $\alpha(k+1) = m+1$, $m = o,1,2,\ldots,2k$, $f^{(m)}(x)$ exists and belongs to $Z_1(a_2,b_2)$,

(4) $\|M_{n,x}^{[k+1]}[f] - f\|_{C[a_3,b_3]} = O(n^{-\alpha(k+1)/2})$.

PROOF First we prove the theorem for $f \in C(\mathbb{R}^+)$. In this case, from Theorem I and relations (3.50), we have the implications (i)$\Rightarrow$(ii)$\Leftrightarrow$(iii)$\Rightarrow$(iv) in the following .

(i) $\|x\{M_{n_p,x}^{[k+1]}[f] - f\}\|_{C[a_1,b_1]} = O(n_p^{-\alpha(k+1)/2})$ $\left(\frac{n_{p+1}}{n_p} \leq c\right)$,

(ii) $x\,f(x) \in \mathrm{Liz}\,(\alpha, k+1, a_2, b_2)$;

(iii)(a) For $m < \alpha(k+1) < m+1$, $m = 0,1,2, \ldots, 2k+2$, $(x\,f(x))^{(m)}$ exists and belongs to $\mathrm{Lip}\,(\alpha(k+1)-m, a_2, b_2)$,

(b) For $\alpha(k+1) = m+1$, $m = 0,1,2, \ldots, 2k$, $(x\,f(x))^{(m)}$ exists and belongs to $Z_1(a_2, b_2)$;

(iv) $\|x\{M_{n,x}^{[k+1]}[f] - f\}\|_{C[a_3,b_3]} = O(n^{-\alpha(k+1)/2})$.

Further, the implications (1)$\Leftrightarrow$(i) and (4)$\Leftrightarrow$(iv) are clear and (2)$\Leftrightarrow$(3) is well known. We now claim that (3)$\Leftrightarrow$(iii). In order to establish this it is sufficient to show that if $0 < a < b < \infty$ then with $0 < \alpha_1 < 1$, the function $(x\,f(x)) \in \mathrm{Lip}\,(\alpha_1, a, b)$ if, and only if $f \in \mathrm{Lip}\,(\alpha_1, a, b)$ and $(x\,f(x)) \in Z_1(a,b)$ if, and only if $f \in Z_1(a,b)$. These results can be easily proved as follows :

Let $f \in \mathrm{Lip}\,(\alpha_1, a, b)$. Then,

$$\begin{aligned} &|(x+h)f(x+h) - xf(x)| \\ &\leq |x\{f(x+h)-f(x)\}| + |hf(x+h)| \\ &\leq b\,M\,h^{\alpha_1} + \max_{x \in [a,b]} |f(x)|\,h, \quad (M \text{ being a Lipschitz constant}) \\ &< M_1\,h^{\alpha_1}, \text{ where } M_1 = \{b\,M + \max_{x\in[a,b]} |f(x)|\,b^{1-\alpha_1}\} \end{aligned}$$

showing that $x\,f(x) \in \mathrm{Lip}\,(\alpha_1, a, b)$.

Next, let $x f(x) \in \text{Lip}\,(\alpha_1, a, b)$. Then,

$$|f(x+h)-f(x)|$$

$$\leq \left| \left\{\frac{1}{x+h} - \frac{1}{x}\right\} \{(x+h)f(x+h)-xf(x)\}\right| + \left|\frac{1}{x} \{(x+h)f(x+h)-xf(x)\}\right| + \left|xf(x) \left\{\frac{1}{x+h} - \frac{1}{x}\right\}\right|$$

$$\leq \frac{b-a}{a^2} M h^{\alpha_1} + \frac{1}{a} M h^{\alpha_1} + \max_{x\in[a,b]} |xf(x)| \frac{h}{a^2},$$

(M being a Lipschitz constant)

$$< M_2 h^{\alpha_1}, \text{ where } M_2 = \left\{M \frac{b-a}{a^2} + \frac{M}{a} + \max_{x\in[a,b]} |xf(x)| \frac{b^{1-\alpha_1}}{a^2}\right\},$$

showing that $f \in \text{Lip}\,(\alpha_1, a, b)$.

Now assume that $f \in Z_1\,(a,b)$ Then

$$\Delta(xf(x)) = \Delta f(x)h + x\Delta f(x) + f(x)h,$$

which implies that

$$\Delta^2(xf(x)) = 2\,\Delta^2 f(x)h + 2\,\Delta f(x)h + x\,\Delta^2 f(x).$$

Hence

$$|\Delta^2(xf(x))| \leq 2(b-a)M h + 2 \max_{x\in[a,b]} \Delta f(x)h + b\,M\,h,$$

(M being a Zygmund constant)

$$\leq M_3\, h, \text{ where } M_3 = \{M(3b-2a) + 4 \max_{x\in[a,b]} |f(x)|\},$$

proving that $x f(x) \in Z_1(a,b)$.

Assuming that $x f(x) \in Z_1(a,b)$, in a similer fashion we can show that $f \in Z_1(a,b)$ and therefore our claim is fully established.

Summarizing, by now we have the following implications .

$$\begin{array}{l} 1 \Leftrightarrow (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Leftrightarrow (4) \\ \qquad\qquad\qquad\qquad\quad\ \Updownarrow \\ \qquad\qquad (2) \Leftrightarrow (3) \end{array}$$

From these $(1) \Rightarrow (2) \Leftrightarrow (3) \Rightarrow (4)$ follow and the proof of the theorem is complete when $f \in C(\mathbb{R}^+)$.

For a general f the proof follows by an observation made at the end of Chapter IV .

When $\alpha = 2$, we have the following saturation theorem for $M_{n,x}^{[k+1]}$, where as in Theorem 3.4.1 the continuity assumption on f is not made.

THEOREM 3.4.2 . Let $f \in B^*$ and $0 < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < \infty$. Then in the following the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) $\Rightarrow$ (6) hold.

(1) $n_p^{k+1} \sup_{x \in [a_1, b_1]} | M_{n_p,x}^{[k+1]} [f(t)] - f(x)| = O(1), (n_{p+1}/n_p \leq c)$,

(2) $f^{(2k+1)} \in A.C. [a_2, b_2]$ and $f^{(2k+2)} \in L_\infty [a_2, b_2]$,

(3) $n^{k+1} || M_{n,x}^{[k+1]} [f] - f ||_{C[a_3, b_3]} = O(1)$,

(4) $n_p^{k+1} \sup_{x \in [a_1, b_1]} | M_{n_p,x}^{[k+1]} [f(t)] - f(x)| = o(1), (n_{p+1}/n_p \leq c)$,

(5) $f \in C^{2k+2} [a_2, b_2]$ and $\sum_{i=k}^{2k+2} Q(i,k,x) f^{(i)} (x) = 0$, $x \in [a_2, b_2]$

where $Q(i,k,x)$ are certain polynomials depending on k ,

(6) $n^{k+1} || M_{n,x}^{[k+1]} [f] - f ||_{C[a_3, b_3]} = o(1)$.

REMARK 3.4.1 : The nature of the polynomials Q(i,k,x) will be made more explict in the proof of the theorem.

The proof of Theorem 3.4.2 given below assumes that $f \in C(\mathbb{R}^+)$. This assumption, as we shall see in the sequel, gets dropped by the observation made at the end of Chapter IV.

PROOF . From (3.50) and Theorem II, in the following the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) hold.

(i) $n_p^{k+1} \| x \{ M_{n_p,x}^{[k+1]} [f] - f \} \|_{C[a_1,b_1]} = O(1), (n_{p+1}/n_p \leq c)$,

(ii) $(xf(x))^{(2k+1)} \in A\,C\,[a_2,b_2]$ and $(xf(x))^{(2k+2)} \in L_\infty [a_2,b_2]$,

(iii) $n^{k+1} \| x \{ M_{n,x}^{[k+1]} [f] - f \} \|_{C[a_3,b_3]} = O(1)$,

(iv) $n_p^{k+1} \| x \{ M_{n_p,x}^{[k+1]} [f] - f \} \|_{C[a_1,b_1]} = o(1), (n_{p+1}/n_p \leq c)$,

(v) $xf(x) \in C^{2k+2} [a_2,b_2]$ and $\sum_{i=k+1}^{2k+2} Q^* (i,k,x)(xf(x))^{(i)} = o, x \in [a_2,b_2]$

where $Q^*(i,k,x)$ are certain polynomials depending on k ,

(vi) $n^{k+1} \| x \{ M_{n,x}^{[k+1]} [f] - f \} \|_{C[a_3,b_3]} = o(1)$.

We further notice that (1) $\Leftrightarrow$ (i), (2) $\Leftrightarrow$ (ii) and (3) $\Leftrightarrow$ (iii) and therefore the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) follow when $f \in C(\mathbb{R}^+)$. Again, the implications (4) $\Leftrightarrow$ (iv), (6) $\Leftrightarrow$ (vi) and the fact that $f \in C^{2k+2} [a_2,b_2]$ if, and only if $(xf(x)) \in C^{2k+2} [a_2,b_2]$ are clear.

Referring to Remark 3.4.1, we actually have

$Q(i,k,x) = x\, Q^*(i,k,x) + (i+1)\, Q^*(i+1,k,x),\ i = k, k+1, \ldots, 2k+2,$

where $Q^*(k,k,x) \equiv o \equiv Q^*(2k+3,k,k)$.

Now assume (4) to be true. Then by (4) $\Leftrightarrow$ (iv) $\Rightarrow$ (v) we have

$$\begin{aligned} o &= \sum_{i=k}^{2k+3} Q^*(i,k,x)\, (xf(x))^{(i)} \\ &= \sum_{i=k}^{2k+3} Q^*(i,k,x) \{ x f^{(i)}(x) + i f^{(i-1)}(x) \} \\ &= \sum_{i=k}^{2k+3} \{ x\, Q^*(i,k,x) + (i+1)\, Q(i+1,k,x) \}\ f^{(i)}(x) \\ &= \sum_{i=k}^{2k+2} Q(i,k,x)\, f^{(i)}(x). \end{aligned}$$

Hence (5) follows i.e ,(4)$\Rightarrow$(5). Next, assume (5), then by the interrelation between the polynomials $Q(i,k,x)$'s and $Q^*(i,k,x)$'s , (V) follows This implies (VI), as remarked before. Now by (VI)$\Leftrightarrow$(6), the proof is complete for $f \in C(\mathbb{R}^+)$

CHAPTER IV

SIMULTANEOUS APPROXIMATION

4.1 INTRODUCTION

In this chapter we study simultaneous approximation properties of the Post-Widder operators $M_{n,x}$. First of all we establish the convergence, $M_{n,x}^{(k)}[f] \to f^{(k)}(x)$ as $n \to \infty$ ($k \in \mathbb{N}$, $x > 0$), whenever $f \in B^*$ and $f^{(k)}(x)$ exists. Next, we obtain asymptotic formulae in simultaneous approximation giving a precise rate of convergence of the derivatives of $M_{n,x}$-operators to the corresponding derivatives of certain smooth functions. We further improve the rate of convergence by considering linear combinations of derivatives of $M_{n,x}$- operators for smoother functions. Finally, we discuss direct, inverse and saturation theorems for $M_{n,x}^{(k)}$ and their linear combinations.

4.2 THE SIMULTANEOUS APPROXIMATION PROPERTY

We start with the following Lorentz-type lemma.

LEMMA 4.2.1 : There exist polynomials $q_{ijk}(x)$ in x which do not depend on t or n such that

(4.1) $$\frac{\partial^k}{\partial x^k}[x^{-(n+1)} e^{-nt/x}] = Q_k(x)\, x^{-n-2k-1} e^{-nt/x},$$

where

(4.2) $$Q_k(x) = \sum_{i,j} n^{i+j} (t-x)^j q_{ijk}(x), \quad i,j \geq 0, \; 2i+j \leq k.$$

PROOF. We have

$$\frac{\partial}{\partial x}\left[x^{-(n+1)} e^{-nt/x}\right] = n\left\{t - \left(\frac{n+1}{n}\right)x\right\} e^{-nt/x} x^{-(n+3)}$$
$$= \{n(t-x)-x\}\, e^{-nt/x} x^{-(n+3)}$$

Hence the lemma is true for $k = 1$. To prove it by induction, we assume the lemma to be true for some k and show that it is true for $(k+1)$.

Now,

$$\frac{\partial^{k+1}}{\partial x^{k+1}}\left[x^{-(n+1)} e^{-nt/x}\right] = Q_k(x)\left[-(n+2k+1)\, x^{-(n+2k+2)} e^{-nt/x} + x^{-(n+2k+3)} nt\, e^{-nt/x}\right]$$
$$+ x^{-(n+2k+1)} e^{-nt/x}\left[\sum_{i,j} n^{i+j}(-j)(t-x)^{j-1} q_{ijk}(x) + \sum_{i,j} n^{i+j} (t-x)^j q'_{ijk}(x)\right]$$
$$= x^{-(n+2k+3)} e^{-nt/x}\left[\{n(t-x)-(2k+1)x\}\, Q_k(x) - \sum_{i,j} n^{i+j}(t-x)^{j-1} j\, x^2 + \sum_{i,j} n^{i+j}(t-x)^j q'_{ijk}(x)x^2\right],$$

which is seen to be of the required form $x^{-(n+2k+3)} e^{-nt/x} Q_{k+1}(x)$, where

$$Q_{k+1}(x) = \sum_{i,j} n^{i+j}(t-x)^j q_{ijk+1}(x), \quad i, j \geq 0, \; 2i+j \leq k+1,$$

with $q_{ijk+1}(x)$ being polynomials in x independent of n and t. This proves the lemma.

In the following theorem, we establish the simultaneous approximation property for the Post-Widder operators.

<u>THEOREM 4.2.1</u> : Let $f \in B^*$ and let $f^{(k)}(x)$ exist for some $x \in \mathbb{R}^+$, then

$$(4.3) \qquad \lim_{n\to\infty} M_{n,x}^{(k)}[f(t)] = f^{(k)}(x).$$

Further, if $f^{(k)}(x)$ exists in $\langle a,b\rangle \subset \mathbb{R}^+$ and is continuous at each $x \in [a,b]$ then (4.3) holds uniformly for x in $[a,b]$.

<u>PROOF</u> : Since $f^{(k)}(x)$ exists,

$$(4.4) \qquad f(t) = \sum_{p=0}^{k} \frac{f^{(p)}(x)}{p!}(t-x)^p + h(t,x),$$

where $h(t,x) \in B^*$ and is such that given an arbitrary $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|h(t,x)| \le \epsilon \, |t-x|^k, \quad |t-x| < \delta.$$

Further, if $f^{(k)}(x)$ exists in $\langle a,b\rangle$ and is continuous at each $x \in [a,b]$, the above δ can be chosen to be independent of x, with the inequality holding for all $x \in [a,b]$.

Now, operating the equation (4.4) by $M_{n,x}^{(k)}$ we have

$$M_{n,x}^{(k)}[f(t)] = \left\{\frac{\partial^k}{\partial y^k} M_{n,y}\left[\sum_{p=0}^{k} \frac{f^{(p)}(x)}{p!}(t-x)^p\right]\right\}_{y=x} + M_{n,x}^{(k)}[h(t,x)]$$

$$= \sum_{p=0}^{k} \frac{f^{(p)}(x)}{p!}\left\{\frac{\partial^k}{\partial y^k} M_{n,y}\left[\sum_{i=0}^{p}\binom{p}{i} t^i(-x)^{p-i}\right]\right\}_{y=x} + M_{n,x}^{(k)}[h(t,x)]$$

$$= \sum_{p=0}^{k} \frac{f^{(p)}(x)}{p!}\left\{\frac{\partial^k}{\partial y^k}\sum_{i=0}^{p}\binom{p}{i}(-x)^{p-i} y^i \frac{\Gamma(n+i+1)}{\Gamma(n+1)n^i}\right\}_{y=x} + M_{n,x}^{(k)}[h(t,x)]$$

(from the formula (1.2)).

Since the first term in the last expression simplifies to $\dfrac{\Gamma(n+k+1)}{\Gamma(n+1)n^k} f^{(k)}(x)$, in order to prove (4.3) it is sufficient to show that

$$(4.5) \qquad \lim_{n\to\infty} M_{n,x}^{(k)}[h(t,x)] = 0,$$

and that (4.5) holds uniformly in the uniformity case.

By Lemma 4.2.1 we have

$$M_{n,x}^{(k)}\ [h(t,x)\chi_{I_\delta(x)}(t)]$$
$$= x^{-2k}\ M_{n,x}[h(t,x)\chi_{I_\delta(x)}(t)\sum_{\substack{i,j\geq 0\\ 2i+j\leq k}} n^{i+j}(t-x)^j q_{ijk}(x)],$$

where $\chi_{I_\delta(x)}$ denotes the characteristic function of the interval $I_\delta(x) = (x-\delta, x+\delta)$.

Now fix a pair (i, j). Then, by the positivity of $M_{n,x}$, in view of (2.3) we have

$$|M_{n,x}\ [h(t,x)(t-x)^j\ \chi_{I_\delta(x)}(t)]|$$
$$\leq \epsilon\ M_{n,x}[|t-x|^{j+k}]$$
$$\leq \epsilon\ M_{j+k}\ n^{-(j+k)/2}\ x^{j+k},$$

for some constant M_{j+k} depending on j and k.

Hence,

$$|\ x^{-2k}\ M_{n,x}\ [h(t,x)\ \chi_{I_\delta(x)}(t)\ n^{i+j}(t-x)^j\ q_{ijk}(x)]|$$
$$\leq \epsilon\ M_{j+k}|q_{ijk}(x)|\ x^{j-k}\ n^{(2i+j-k)/2}.$$

Since $2i+j \leq k$, it follows that there exists a constant $C(x,k)$, not depending on n or ϵ, such that

(4.6) $$|M_{n,x}^{(k)}\ [h(t,x)\ \chi_{I_\delta(x)}(t)]| < \epsilon\ C\ (x,k) \text{ for all } n \geq 1.$$

Now, with $J_\delta(x) = (0,\infty)\setminus I_\delta(x)$, by Lemma 4.2.1,

$$M_{n,x}^{(k)}\ [h(t,x)\ \chi_{J_\delta(x)}(t)]$$
$$= \sum_{\substack{i,j\geq 0\\ 2i+j\leq k}} n^{i+j}\ q_{ijk}(x)\ x^{-2k}\ M_{n,x}[h(t,x)\chi_{J_\delta(x)}(t)\ (t-x)^j],$$

which by Lemma 1.4.3 is seen to be of $o(n^{-m})$ for each $m>0$ as $n \to \infty$.

Hence,

(4.7) $$\lim_{n\to\infty} M_{n,x}^{(k)}[\ h(t,x)\chi_{J_\delta(x)}(t)] = 0.$$

From (4.6) and (4.7), due to the arbitrariness of ϵ, we have (4.5). Further, (4.5) holds uniformly in the uniformity case. This completes the proof of the theorem.

4.3 ASYMPTOTIC FORMULAE IN THE SIMULTANEOUS APPROXIMATION

The following theorem gives an asymptotic formula of the type obtained by Rathore [71,72] in simultaneous approximation.

THEOREM 4.3.1 . If $f \in B^*$ and $f^{(k+2)}(x)$ exists at a point $x \in (0,\infty)$, then

$$(4.8)\quad M_{n,x}^{(k)}[f(t)] - f^{(k)}(x) = \frac{1}{2n}[f^{(k)}(x)k(k+1) + 2f^{(k+1)}(x)(k+1)x + f^{(k+2)}(x)x^2] + o(n^{-1}), n \to \infty .$$

Further, if $f^{(k+2)}(x)$ exists in $\langle a,b\rangle \subset \mathbb{R}^+$ and is continuous at each $x \in [a,b]$ then (4.8) holds uniformly in $x \in [a,b]$.

PROOF : Taking $m = 1$ in the subsequent Theorem 4.3.2, Theorem 4.3.1 follows from (1.9) after simple computations.

The following theorem gives a generalized asymptotic formula in the simultaneous approximation and shall be used in the next section in obtaining the order of simultaneous approximation by linear combinations of $M_{n,x}^{(k)}$.

THEOREM 4.3.2 . If $f \in B^*$ and $f^{(2m+k)}(x)$ exists at a point $x \in (0,\infty)$, then

$$(4.9)\quad M_{n,x}^{(k)}[f(t)] = \frac{\partial^k}{\partial x^k}\left[\sum_{p=0}^{2m} \frac{f^{(p)}(x)}{p!}\mu_{n,p}(x)\right] + o(n^{-m}), n\to\infty .$$

Further, if $f^{(2m+k)}(x)$ exists in $\langle a,b\rangle \subset \mathbb{R}^+$ and is continuous at each $x \in [a,b]$, than (4.9) holds uniformly in $x \in [a,b]$.

PROOF : If $f \in B^*$ and $f^{(2m+k)}(x)$ exists, then

(4.10) $$f(t) = \sum_{p=0}^{2m+k} \frac{f^{(p)}(x)}{p!} (t-x)^p + h(t,x),$$

where given an arbitrary $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|h(t,x)| \leq \epsilon |t-x|^{2m+k}, \quad |t-x| < \delta.$$

If $f^{(2m+k)}(x)$ exists in $\langle a,b\rangle$ and is continuous at each $x \in [a,b]$, δ can be chosen independently for each $x \in [a,b]$.

First, we shall show that

(4.11) $$M_{n,x}^{(k)} [h(t,x)] = o(n^{-m}), \quad n \to \infty,$$

and that in the above mentioned uniformity case, (4.11) holds uniformly for each $x \in [a,b]$.

With $I_\delta(x)$, $J_\delta(x)$ as in the proof of Theorem 4.2.1, we have

$$M_{n,x}^{(k)} [h(t,x)] = \{ M_{n,y}^{(k)} [h(t,x)]\}_{y=x}$$

(4.12) $$= \left\{\frac{\partial^k}{\partial y^k} \left[\frac{n^{n+1}}{y^{n+1}\Gamma(n+1)} \int_{I_\delta(x)} h(t,x) t^n e^{-nt/y} dt \right]\right\}_{y=x}$$

$$+ \left\{\frac{\partial^k}{\partial y^k} \left[\frac{n^{n+1}}{y^{n+1}\Gamma(n+1)} \int_{J_\delta(x)} h(t,x) t^n e^{-nt/y} dt\right]\right\}_{y=x}.$$

Now, by Lemma 4.2.1,

$$\left\{\frac{\partial^k}{\partial y^k} \left[\frac{n^{n+1}}{y^{n+1}\Gamma(n+1)} \int_{I_\delta(x)} h(t,x) t^n e^{-nt/y} dt\right] \right\}_{y=x}$$

$$= \frac{n^n}{\Gamma(n)} x^{-n-2k-1} \sum_{\substack{i,j\geq 0 \\ 2i+j\leq k}} n^{i+j} \int_{I_\delta(x)} (t-x)^j q_{ijk}(x) h(t,x) t^n e^{-nt/x} dt.$$

Considering a general term in the summation on the right hand side, we have

$$\left| \frac{n^n}{\Gamma(n)} x^{-n-2k-1} n^{i+j} q_{ijk}(x) \int_{I_\delta(x)} (t-x)^j h(t,x) t^n e^{-nt/x} dt \right|$$

$$\leq \frac{n^n}{\Gamma(n)} x^{-n-2k-1} n^{i+j} |q_{ijk}(x)| \int_{I_\delta(x)} |t-x|^j |h(t,x)| t^n e^{-nt/x} dt$$

$$\leq \epsilon \frac{n^n}{\Gamma(n)} x^{-2k} x^{-n-1} n^{i+j} | q_{ijk}(x) | \int_{I_\delta(x)} |t-x|^{2m+k+j} t^n e^{-nt/x} dt$$

$$\leq \epsilon A_{2m+j+k} x^{-2k} n^{i+j} |q_{ijk}(x)| n^{-(2m+k+j)/2},$$

(by the asymptotic evaluation(2.3), where A_{2m+j+k} is a constant depending on $2m+j+k$)

$$\leq \epsilon A_{2m+j+k} x^{-2k} | q_{ijk}(x)| n^{-m}, \quad (n \geq 1, \text{ since } (k+j)/2 \geq i+j).$$

As $\epsilon > 0$ is arbitrary, we have

$$(4.13) \quad \left\{ \frac{\partial^k}{\partial y^k} \left[\frac{n^{n+1}}{y^{n+1}\Gamma(n+1)} \int_{I_\delta(x)} h(t,x) t^n e^{-nt/y} dt \right] \right\}_{y=x} = o(n^{-m}), n\to\infty$$

Further, (4.13) holds uniformly in $x \in [a,b]$ in the uniformity case.

Also, by Lemmas 4.2.1 and 1.4.3,

$$(4.14) \quad \left\{ \frac{\partial}{\partial y^k} \left[\frac{n^{n+1}}{y^{n+1}\Gamma(n+1)} \int_{I_\delta'(x)} h(t,x) t^n e^{-nt/y} dt \right] \right\}_{y=x} = o(n^{-s}), n\to\infty,$$

for an arbitrary $s > 0$ (and (4.14) holds uniformly in $x \in [a,b]$ in the uniformity case).

Equations(4.13) and (4.14) establish (4.11).

In view of (4.11), to prove the theorem it is sufficient to show that

$$M_{n,x}^{(k)} \left[\sum_{p=0}^{2m+k} \frac{f^{(p)}(x)}{p!} (t-x)^p \right]$$

$$(4.15) \quad = \frac{\partial^k}{\partial x^k} \left[\sum_{p=0}^{2m} \frac{f^{(p)}(x)}{p!} \mu_{n,p}(x) \right] + o(n^{-m}),$$

and that it holds uniformly in $x \in [a,b]$ in the uniformity case

For this, by the generalized asymptotic formula in ordinary approximation, if $Q(t)$ is a polynomial in t,

(4.16) $$M_{n,x}[Q(t)] = \sum_{p=0}^{2m} \frac{Q^{(p)}(x)}{p!}\,\mu_{n,p}(x) + o(n^{-m}),\ n \to \infty,$$

where the relation is also uniform in $x \in [a,b]$. By (1.6) it follows that $M_{n,x}[Q(t)]$ is also a polynomial of the same degree as $Q(t)$ and that the small order term on the right hand side of (4.16) consists of a finite sum of terms like $\frac{P_{m+1}(x)}{n^{m+1}}$, $\frac{P_{m+2}(x)}{n^{m+2}}$, . . ., where $P_{m+1}(x)$, $P_{m+2}(x)$ are polynomials in x. Therefore,

(4.17) $$M^{(k)}_{n,x}[Q(t)] = \frac{\partial^k}{\partial x^k}\Big[\sum_{p=0}^{2m} \frac{Q^{(p)}(x)}{p!}\,\mu_{n,p}(x)\Big] + o(n^{-m}),\ n \to \infty,$$

which also holds uniformly in $x \in [a,b]$. Taking

$$Q(t) = \sum_{i=0}^{2m+k} \frac{f^{(i)}(x)}{i!}(t-x)^i \text{ in (4.17),}$$

and keeping in mind the boundedness of the derivatives of f we, thus, get (4.15) and also that it holds uniformly in the uniformity case. This completes the proof of the theorem.

4.4 LINEAR COMBINATIONS AND SIMULTANEOUS APPROXIMATION

In this section we consider the linear combinations $M^{(k)[m]}_{n,x}$ of the operators $M^{(k)}_{n,x}$ defined by replacing $M_{\alpha_i n,x}[f(t)]$ in (1.31) by $M^{(k)}_{\alpha_i n,x}[f(t)]$, $i = 0,1,\ldots m-1$. Obviously, there holds the interrelation $M^{[m](k)}_{n,x}[f] = M^{(k)[m]}_{n,x}[f]$.

By Theorem 4.3.1 it is clear that a smoothness of f beyond the existence of $f^{(k+2)}(x)$ does not result in an improved approximation of $f^{(k)}(x)$ by $M^{(k)}_{n,x}[f]$. Regarding the linear combinations $M^{(k)[m]}_{n,x}$, however, we have

THEOREM 4.4.1 : Let $f \in B^*$ and let $f^{(2m+k)}(x)$ exist at a point $x \in \mathbb{R}^+$ then

(4.18) $$M_{n,x}^{(k)[m+1]}[f(t)] - f^{(k)}(x) = o(n^{-m}) \text{ as } n \to \infty,$$

and

(4.19) $$M_{n,x}^{(k)[m]}[f(t)] - f^{(k)}(x) = O(n^{-m}) \text{ as } n \to \infty$$

Further, if $f^{(2m+k)}$ exists on $\langle a,b \rangle \subset \mathbb{R}^+$ and is continuous at each $x \in [a,b]$, then (4.18) - (4.19) hold uniformly in $x \in [a,b]$.

The proof of Theorem 4.4.1 can be easily obtained by using Theorem 4.3.2 and proceeding as in the proof of Theorem 1.6.1.

THEOREM 4.4.2 : Let $k \in \mathbb{N}$ and m be a non-negative integer. If $f \in B^*$ and $f^{(2m+k)}(x)$ exists and is continuous on $\langle a,b \rangle \subset \mathbb{R}^+$ then

(4.20) $$|M_{n,x}^{(k)[m+1]}[f(t)] - f^{(k)}(x)| \leq \max\left\{\frac{C}{n^m}\,\omega(f^{(2m+k)}, n^{-1/2}), \frac{C'}{n^{m+1}}\right\},$$
$$x \in [a,b], \text{ where } C = C(m) \text{ and } C' = C'(m,f).$$

PROOF : With the hypothesis on f, for all $x \in [a,b]$, we can write

(4.21) $$f(t) = \sum_{i=0}^{2m+k} \frac{f^{(i)}(x)}{i!}(t-x)^i + \frac{(t-x)^{2m+k}}{(2m+k)!}\{f^{(2m+k)}(\eta) - f^{(2m+k)}(x)\}\chi_{\langle a,b\rangle}(t)$$
$$+(1-\chi_{\langle a,b\rangle}(t))\, h(t,x),\ t > 0,$$

where $\chi_{\langle a,b\rangle}$ is the characteristic function of $\langle a,b\rangle$, η lies between t and x and h(t,x) is a certain function belonging to B^*

By the definition of $M_{n,x}^{(k)[m]}[f(t)]$, we have

$$M_{n,x}^{(k)[m+1]}[f(t)] - f^{(k)}(x) = \sum_{j=0}^{m}\{B_j\,[M_{\alpha_j n,x}^{(k)}[f(t)] - f^{(k)}(x)]\},$$
(since $B_0+B_1\ldots B_m = 1$).

$$=\sum_{j=0}^{m}\{B_j(M_{\alpha_j n,x}^{(k)}[\sum_{i=0}^{2m+k}\frac{f^{(i)}(x)}{i!}(t-x)^i+\frac{(t-x)^{2m+k}}{(2m+k)!}(f^{(2m+k)}(\eta)-f^{(2m+k)}(x))\chi_{\langle a,b\rangle}(t)+(1-\chi_{\langle a,b\rangle}(t))h(t,x)]-f^{(k)}(x)\})\}.$$

Thus, defining

$$\tau_{x,\alpha_j n}(t)=\left(\frac{\alpha_j n}{x}\right)^{\alpha_j n+1}\frac{1}{\Gamma(\alpha_j n+1)}\,t^{\alpha_j n}e^{-\alpha_j nt/x}Q_{k,j}(x)x^{-\alpha_j n-2k-1},$$

where

$$Q_{k,j}(x)=\sum_{\substack{i_1,i_2\geq 0\\ 2i_1+i_2\leq k}}(\alpha_j n)^{i_1+i_2}(t-x)^{i_2}q_{i_1i_2k}(x),$$

by Lemma 4.2.1, we have

$$M_{n,x}^{(k)}[m+1][f(t)]-f^{(k)}(x)=\{M_{n,x}^{(k)}[m+1][\sum_{i=0}^{2m+k}\frac{f^{(i)}(x)}{i!}(t-x)^i]-f^{(k)}(x)\}$$

$$+\sum_{j=0}^{m}B_j\int_0^\infty\frac{(t-x)^{2m+k}}{(2m+k)!}[f^{(2m+k)}(\eta)-f^{(2m+k)}(x)]\tau_{x,\alpha_j n}(t)\chi_{\langle a,b\rangle}(t)\,dt$$

$$+\sum_{j=0}^{m}B_j\int_0^\infty h(t;x)(1-\chi_{\langle a,b\rangle}(t))\tau_{x,\alpha_j n}(t)\,dt \tag{4.22}$$

$$=\Sigma_1+\Sigma_2+\Sigma_3,\ \text{say}$$

In view of Theorem 4.4.1 (we use (4.19) with m replaced by m+1), it is clear that

$$|\Sigma_1|\leq C_1n^{-m-1}, \tag{4.23}$$

where C_1 depends on maximum modulii of various derivatives $f^{(i)}(x)$ on $[a,b]$ and is independent of n.

To evaluate Σ_2 we proceed as follows :

$$\int_0^\infty\frac{|t-x|^{2m+k}}{(2m+k)!}|f^{(2m+k)}(\eta)-f^{(2m+k)}(x)|\chi_{\langle a,b\rangle}(t)|\tau_{x,\alpha_j n}(t)|\,dt$$

$$\leq\frac{\omega(f^{(2m+k)};\delta)}{(2m+k)!}\{\int_0^\infty|t-x|^{2m+k}|\tau_{x,\alpha_j n}(t)|dt+\frac{1}{\delta}\int_0^\infty|t-x|^{2m+k+1}|\tau_{x,\alpha_j n}(t)|dt\}$$

Using the asymptotic evaluation (2.3), an analysis similar to that of the proof of Theorem 4.3.2 shows that the last expression does not exceed

$$\frac{\omega(f^{(2m+k)},\delta)}{(2m+k)!}\left\{\frac{A_k}{(\alpha_j n)^m}+\frac{A_k'}{\delta(\alpha_j n)^{m+1/2}}\right\},$$

where A_k, A_k' are constants independent of $\alpha_j n$ and f.

Therefore, with $\delta = n^{-1/2}$, we have

(4.24) $\quad |\Sigma_2| \leq \dfrac{C_2}{n^m} \omega(f^{(2m+k)},n^{-1/2}),$

for some constant C_2 not depending on n.

Finally, in view of Lemma 1.4.3, it is clear that $\Sigma_3 = o(n^{-s})$ uniformly in $x \in [a,b]$ for an arbitrary $s > 0$

Hence there exists a constant C_3 depending on f but independent of n such that

(4.25) $\quad |\Sigma_3| \leq \dfrac{C_3}{n^{m+1}}.$

The estimates (4.23) - (4.25) prove the theorem.

In a similar fashion we can also prove the following theorem.

<u>THEOREM 4.4.3</u> : If $f \in B^*$ and $f^{(2m+k+1)}(x)$ exists and is continuous on $\langle a,b\rangle$ then

(4.26) $\quad |M_{n,x}^{(k)[m+1]}[f(t)] - f(x)| < \max\left\{\dfrac{C}{n^{m+1/2}}\,\omega(f^{(2m+k+1)},n^{-1/2}), \dfrac{C'}{n^{m+1}}\right\},$

$x \in [a,b]$, where $C = C(m)$ and $C' = C'(m,f)$.

4.5 DIRECT, INVERSE AND SATURATION THEOREMS IN THE SIMULTANEOUS APPROXIMATION

In this section we obtain direct, inverse and saturation theorems in the simultaneous approximation. We shall

ee how the direct, inverse and saturation theorems in the rdinary approximation can be extended to the simultaneous approximation case. We start with

THEOREM 4.5.1 : Let f be such that $f^{(k)}$ exists and belongs to B^* then

(4.27) $\quad |M_{n,x}^{(k)}[f(t)] - f^{(k)}(x)| \leq \frac{M}{2n} x^{-k+1}$, for all n sufficiently large and $x \in \mathbb{R}^+$ if, and only if $\{x^{k+1} f^{(k)}(x)\} \in Lip_M 1$.

PROOF : We have

$$M_{n,x}^{(k)}[f(t)] = \frac{n^n}{\Gamma(n)} \int_0^\infty t^k f^{(k)}(tx) t^n e^{-nt}\, dt$$

$$(4.28) \qquad = \frac{n^n}{\Gamma(n)x^k} \int_0^\infty (tx)^k f^{(k)}(tx)\, t^n e^{-nt}\, dt$$

$$= \frac{1}{x^k} M_{n,x}[t^k f^{(k)}(t)] .$$

The theorem now follows from Theorem 3.2.1.

In a similar manner, using Theorems 3.2.2 and 3.2.3 we can prove the following theorems.

THEOREM 4.5.2 : Let $f^{(k)}$ exist and belong to B^*. If $0 < \alpha \leq 2$ and $|f^{(k)}(t)| \leq M t^{\alpha-k-1}$ for some constant M and all $t \in \mathbb{R}^+$, then the following statements are equivalent:

(i) $\quad \{x^{k+1} f^{(k)}(x)\} \in Z_\alpha^*$,

(ii) $\quad$ for some constant $A > 0$, and all n sufficiently large,

(4.29) $\quad |M_{n,x}^{(k)}[f(t)] - f^{(k)}(x)| \leq A\, x^{\alpha-k-1} n^{-\alpha/2}$ $(x \in \mathbb{R}^+)$.

THEOREM 4.5.3 : Let $f^{(k)}$ exist and belong to B^* and for some α satisfying $0 < \alpha \leq 2$, $|f^{(k)}(t)| \leq M t^{\alpha-k-1}$ for some constant M and all $t \in \mathbb{R}^+$. Then the following statements are equivalent .

(i) $\{x^{k+1} f^{(k)}(x)\} \in Z_{\alpha}$,

(ii) for some constant $A > o$ and all n sufficiently large,
$$|M_{n,x}^{(k)} [f(t)] - f^{(k)}(x)| \leq A\, x^{\alpha-k-1} n^{-\alpha/2} \quad (x \in \mathbb{R}^+)$$

The previous theorems of this section were of a global nature and therefore it was necessary to make the assumptions regargarding the existence and growth of $f^{(k)}$ on the whole of $\mathbb{R}^+$. The following results, however, are of a local character in which it is possible to do away with similar assumptions.

<u>THEOREM 4.5.4</u> : Let $f \in B^*$, $o < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < \infty$ and $o < \alpha < 2$. Then in the following the implications (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4) hold, provided (1) is meaningful i.e., $f^{(k)}(x)$ exists on $[a, b]$.

(1) $\sup\limits_{x \in [a_1, b_1]} |M_{n_p,x}^{(k)[m+1]} [f(t)] - f^{(k)}(x)| = O(n_p^{-\alpha(m+1)/2})$, $\left(\frac{n_{p+1}}{n_p} \leq c\right.$, for some constant $c)$,

(2) $f^{(k)}(x) \in \text{Liz}\,(\alpha, m+1, a_2, b_2)$,

(3) (i) for $m' < \alpha(m+1) < m'+1$, $m' = o, 1, 2, \ldots, 2m+1$,
$f^{(k+m')}(x)$ exists and belongs to $\text{Lip}\,(\alpha(m+1)-m', a_2, b_2)$,
(ii) for $\alpha(m+1) = m'+1$, $m' = o, 1, 2, \ \ , 2m$, $f^{(k+m')}$ exists and belongs to $Z_1(a_2, b_2)$,

(4) $\| M_{n,x}^{(k)[m+1]} [f] - f^{(k)} \|_{C[a_3, b_3]} = O(n^{-\alpha(m+1)/2})$.

<u>PROOF</u> : Assume (1). Since $M_{n,x}^{(k)[m+1]}$ are continuous functions and by (1) they converge to $f^{(k)}(x)$ uniformly on $[a_1, b_1]$, it follows that $f^{(k)}(x)$ is continuous in the relative topology of $[a_1, b_1]$. Let a_1', b_1', a_1'' and b_1'' satisfy $a_1 < a_1'' < a_1' < a_2$, $b_2 < b_1' < b_1'' < b_1$. Let $f_* \in C_o^k(\mathbb{R}^+)$ with supp $f_* \subset (a_1, b_1)$

such that $f_*(x) = f(x)$ for $x \in (a_1'', b_1'')$. The existence of such a function f_* is trivial. Now, in view of Lemmas 1.4.3 and 1.2.1 it is clear that

$$\|M_{n,x}^{(r)\,[r+1]}\,[f - f_*]\|_{C[a_1', b_1']} = o\,(n^{-s}),$$

for an arbitrary $s > o$.

It follows that

$$\|M_{n,x}^{(k)\,[r+1]}\,[f_*] - f_*\|_{C[a_1', b_1']} = O(n^{-\alpha(m+1)/2}).$$

In view of the validity of (4.28) for f_*, Theorem 4.5.4 follows from the continuous version of Theorem 3.4.1 and the intermediate steps in its proof.

By arguments similar to those considered for the proof of Theorem 4.5.4, from the continuous version of Theorem 3.1.2 and its proof there follows the saturation

<u>THEOREM 4.5.5</u> . Let $f \in B^*$ and $o < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < \infty$ Then in the following (1) $\Rightarrow$(2) $\Rightarrow$(3) and (4)$\Rightarrow$(5)$\Rightarrow$(6) hold, provided (1) is meaningful. i.e., $f^{(k)}(x)$ exists on $[a_1, b_1]$

(1) $n_p^{m+1} \sup\limits_{x \in [a_1, b_1]} |M_{n_p,x}^{(k)\,[m+1]}\,[f(t)] - f^{(k)}(x)| = O(1)$
($\frac{n_{p+1}}{n_p} \leq c$ where c is a constat);

(2) $f^{(k+2m+1)} \in A.C.\,[a_2, b_2]$ and $f^{(k+2m+2)} \in L_\infty\,[a_2, b_2]$,

(3) $n^{m+1}\,\|M_{n,x}^{(k)\,[m+1]}\,[f] - f^{(k)}\|_{C\,[a_3, b_3]} = O(1)$,

(4) $n_p^{m+1} \sup\limits_{x \in [a_1, b_1]} |M_{n_p,x}^{(k)\,[m+1]}[f(t)] - f^{(k)}(x)| = o(1)$,

(5) $f^{(k)}(x) \in C^{2m+2}\,[a_2, b_2]$ and $\sum\limits_{i=\max(m,k)}^{2m+2+k} Q(i,m,x) f^{(i)}(x) = o$,

$x \in [a_2, b_2]$, where $Q(i,m,x)$ are certain polynomials depending on m and k ;

(6). $n^{m+1} \; || M_{n,x}^{(k)[m+1]}[f] - f^{(k)} ||_{C[a_3,b_3]} = o(1).$

Finally , we observe that to complete the proofs of Theorems 3.4.1 and 3.4.2 for a general function f, it is sufficient to work with an auxiliary function $f_* \in C_0^k(\mathbb{R}^+)$ with supp $f_* \subset (a_1, b_1)$ and coinciding with f on (a_1'', b_1'') and proceeding in the manner of the proof of Theorem 4.5.4.

Chapter V

ANALYTIC FUNCTIONS AND POST-WIDDER OPERATORS

5 1 INTRODUCTION

So far the behaviour of very few linear operators defined with the help of function values on the real line or a subset of it has been considered in the complex plane The first such nontrivial operators whose behaviour in the complex plane was studied were the Bernstein polynomials, and the study was done by Wright [91] and Kantorovitch [30] . Kantorovitch proved the following important result regarding the convergence of Bernstein polynomials

$B_n^f(z) = \sum_{r=o}^{n} f(\frac{r}{n})\binom{n}{r}\ z^n(1-z)^{n-1}$ for complex values of z lying outside the segment $o \leq z \leq 1$

THEOREM I If f(z) is analytic in the interior E of the ellipse with foci at o, 1, then $B_n^f(z) \to f(z)$ as $n \to \infty$ on E, this convergence being uniform on each closed subset of E.

Kantorovitch's proof of the above theorem is based on an expansion of analytic functions in a series of Legendre polynomials. Bernstein earlier proved a similar result for circles instead of ellipses. His further contributions [4-6] also contain some refinements over Theorem I. A comprehensive account of this work on Bernstein polynomials can be found in Lorentz [42] .

Cheney and Sharma [14] have applied a similar technique to study other sequences of linear positive operators in the complex plane. J J Gergen, F.G Dressel, and W H.Purcell, Jr [22] have studied the problem of convergence of Szász operators $P_k(z,f) = e^{-kz} \sum_{n=0}^{\infty} \frac{(kz)^n}{n!} f(\frac{n}{k})$, k in the complex plane. They obtained an analogue of Theorem I for these operators. The sets of convergence, in this case, are parabolas. Their method of proof is based on the fact that if f is analytic in a certain parabolic set and satisfies a certain growth restriction then it possesses a Laguerre series expansion.

The object of this chapter is to discuss the behaviour of $M_{n,z}[f]$ for complex z lying in the open right half plane which we shall denote by $\mathbb{C}^+$ We shall be interested in an analogue of Theorem I for these operators, namely, in the convergence,

$$\lim_{n \to \infty} M_{n,z}[f(t)] = f(z), \tag{5 1}$$

for appropriate $z \in \mathbb{C}^+$ and analytic functions f whose restrictions to $\mathbb{R}^+$ belong to B^*, and which for some constants $B, b > 0$ satisfy the growth restriction

$$\begin{aligned} f(z) &= O(e^{B|z|}), \quad |z| \to \infty \text{ and} \\ f(z) &= O(|z|^{-b}), \quad |z| \to 0, \ z \in \mathbb{C}^+, \end{aligned} \tag{5 2}$$

uniformly for arg(z) belonging to compact subsets of $(-\pi/2, \pi/2)$

We also discuss the effect of singularities of f on the region of convergence and obtain best possible regions of convergence corresponding to locations of singularities.

5.2 CHARACTERIZATION OF CERTAIN CLASSES OF ANALYTIC FUNCTIC

The following theorem characterises certain classes of analytic functions. At the end of this chapter, we shall use this theorem to establish some results regarding convergence of Post-Widder operators in $\mathbb{C}^+$.

THEOREM 5.2.1 Let $f \in B^*$ and let $\mathbb{N}_{f,x} (x \in \mathbb{R}^+)$ denote the set of all n such that $M_{n,x}[f(t)]$ exists. Then, if $f \in C(\mathbb{R}^+)$ and $\theta \in (0, \pi/2)$, the following statements are equivalent:

(i) f has an extension as an analytic function regular in the set $K(\theta) = \{ z | \operatorname{Re} z > 0, \; |\arg z| < \theta \}$ and $\lim_{n \to \infty} M_{n,z}[f(t)] = f(z)$, uniformly on each compact subset of $K(\theta)$.

(ii) For each $x \in \mathbb{R}^+$ there exists a positive real number M_x such that

$$(5.3) \qquad k^{-1} \left| M_{n,x}^{(k)}[f(t)] \right|^{1/k} < M_x, \; k \in \mathbb{N}, \; n \in \mathbb{N}_{f,\bar{x}}$$

and

$$(5.4) \qquad \limsup_{k \to \infty} k^{-1} \left| M_{n,x}^{(k)}[f(t)] \right|^{1/k} \leq (e \, x \sin\theta)^{-1}$$

uniformly for $n \in \mathbb{N}_{f,\bar{x}}$, where $\bar{x} = x(1+\sin\theta)$.

PROOF First assume (ii) to be true. Then as $\mathbb{N}_{f,\bar{x}} \subseteq \mathbb{N}_{f,x}$, for each $n \in \mathbb{N}_{f,\bar{x}}$ the function

$$(5.5) \qquad g_n(z) = \sum_{k=0}^{\infty} \frac{(z-x)^k}{k!} M_{n,x}^{(k)}[f(t)]$$

is an analytic function regular in $S_{x, x\sin\theta} = \{ z | \, |z-x| < x\sin\theta \}$ as the radius of convergence of the right hand side series

of (5.5) is atleast $x\sin\theta$. Since $f \in C(R^+)$, from Theorem 1 there exists a constant M_x' such that $|M_{n,x}[f(t)]| < M_x', n \in \mathbb{N}_f$. Now, in view of the Stirling's formula and (5.4), given an arbitrary $\gamma > 1$ there exists a $k_o \in \mathbb{N}$ such that

$$(5.6) \qquad (k!)^{-1} |M_{n,x}^{(k)}[f(t)]| < \left(\frac{\gamma}{x\sin\theta}\right)^k, n \in \mathbb{N}_{f,\bar{x}}, \quad k > k_o.$$

Hence if $\lambda, \rho > o$ are such that $\lambda\gamma = \rho < 1$, we have for $z \in S_{x, \lambda x\sin\theta}$

$$|g_n(z)| \leq \sum_{k=o}^{k_o} \left|\frac{(z-x)^k}{k!} M_{n,x}^{(k)}[f(t)]\right| + \sum_{k=k_o+1}^{\infty} \left|\frac{(z-x)^k}{k!} M_{n,x}^{(k)}[f(t)]\right|$$

$$(5.7) \qquad \leq M_x' + \sum_{k=1}^{k_o} \frac{(\lambda x\sin\theta)^k}{k!} (kM_x)^k + \frac{\rho^{k_o+1}}{1-\rho}, n \in \mathbb{N}_{f,\bar{x}}$$

It follows that $\{g_n(z)\}_{n \in \mathbb{N}_{f,\bar{x}}}$ is a net of analytic functions regular and uniformly bounded in $S_{x, \lambda x\sin\theta}$. Since for $n \in \mathbb{N}_{f,\bar{x}}$, $M_{n,z}[f(t)]$ is an analytic function regular in $S_{x, x\sin\theta}$, $g_n(z) = M_{n,z}[f(t)]$ inside the circle of convergence. In view of the continuity of $f(x)$ and Theorem 1.5.1 Vitali's theorem implies that $f(x)$ has an extension as an analytic function regular in $S_{x, x\sin\theta}$ and that $M_{n,z}[f(t)] \to f(z)$ as $n \to \infty$ uniformly on each compact subset of $S_{x, \lambda x\sin\theta}$.

Now choosing ρ and γ sufficiently near to 1, we can make λ as close to 1 as we please. Therefore, the extension $f(z)$ of $f(x)$ is an analytic function regular inside $S_{x, \lambda x\sin\theta}$ and $M_{n,z}[f(t)] \to f(z)$ as $n \to \infty$ on

each compact subset of $S_{x,\ \lambda x\sin\theta}$. This being true for each $x \in \mathbb{R}^+$, in view of the fact that each compact subset of $K(\theta)$ can be covered with a finite number of sets of the type $S_{x,\ \lambda\sin\theta}(\lambda < 1)$, (i) follows.

Next, assume (i) to be true. Then the Taylor series for $f(z)$ about x has radius of convergence atleast equal to $x\sin\theta$. Consider a circle Γ of radius $\lambda x\sin\theta$ and centre x, $(0 < \lambda < 1)$. By Cauchy's inequality we can write

$$|f^{(k)}(x)| \leq \frac{k!\, M_1}{(\lambda x\sin\theta)^k}, \quad k \in \mathbb{N},$$

where M_1 is the maximum modulus of $f(z)$ on Γ. Applying Cauchy's integral formula we have

$$(5.8) \quad |M^{(k)}_{n,x}[f(t)] - f^{(k)}(x)| \leq \frac{k!}{2\pi} \int_\Gamma \frac{|M_{n,z}[f(t)] - f(z)|}{|z-x|^{k+1}} |dz|$$

But, from the uniform convergence of $M_{n,z}[f(t)]$ to $f(z)$ as $n \to \infty$ on each compact subset of $K(\theta)$, we get

$$|M_{n,z}[f(t)] - f(z)| < M_2 \text{ for all } n \in \mathbb{N}_{f,\bar{x}},\ z \in \Gamma$$

and some constant M_2

Therefore,

$$(5.9) \quad |M^{(k)}_{n,x}[f(t)] - f^{(k)}(x)| \leq \frac{k!\ M_2\ 2\pi\ \lambda\ x\sin\theta}{2\pi(\lambda\ x\sin\theta)^{k+1}}$$

$$= \frac{k!\, M_2}{(\lambda\ x\sin\theta)^k}$$

for all $k \in \mathbb{N}$ and $n \in \mathbb{N}_{f,\bar{x}}$.

From (5.8) and (5.9), we have

$$|M_{n,x}^{(k)}\ [f(t)]| \leq \frac{k!}{(\lambda\, x\sin\theta)^k}\{M_1 + M_2\}\ ,$$

for all $k \in \mathbb{N}$ and $n \in \mathbb{N}_{f,\bar{x}}$, whence

$$k^{-1}|M_{n,x}^{(k)}[f(t)]\ |^{1/k} \leq \frac{(k!)^{1/k}/k}{\lambda\, x\sin\theta}\ (M_1 + M_2\ )^{1/k}\ ,$$

for all $k \in \mathbb{N}$ and $n \in \mathbb{N}_{f,\bar{x}}$.

In view of the Stirling's formula there exists a constant C such that for all $k \in \mathbb{N}$, $((k!)^{1/k}/k)\ (M_1+M_2) < C$. Then, with $M_x = \dfrac{C}{\lambda\, x\sin\theta}$, we have (5.3), proving the first part of (ii). Further,

$$\limsup_{k \to \infty} k^{-1}|M_{n,x}^{(k)}[f(t)]|^{1/k} \leq \lim_{k \to \infty} \frac{(k!)^{1/k}/k}{\lambda\, x\sin\theta}\ (M_1+M_2)^{1/k}$$

$$= \frac{1}{e\ \lambda x\sin\theta}\ \text{(uniformly in } n \in \mathbb{N}_{f,\bar{x}}).$$

But, the left hand side in the above inequality is independent of λ . Hence,

$$\limsup_{k \to \infty} k^{-1}\ |M_{n,x}^{(k)}[f(t)]\ |^{1/k} \leq (e x\sin\theta)^{-1},$$

(uniformly in $n \in \mathbb{N}_{f,\bar{x}}$), which completes the proof of the implication (i) $\Rightarrow$ (ii), thus proving the theorem.

<u>REMARK 5.2.1</u> From the above proof of (ii) $\Rightarrow$ (i) it is clear that when the hypothesis (ii) of Theorem 5.2.1 is assumed only for an unbounded subset $\mathbb{N}_{f,x}^{o}$ (of $\mathbb{N}_{f,x}$) $\supseteq \mathbb{N}_{f,\bar{x}}$, even then (i) follows with the convergence $M_{n,z}\ f(t) \to f(z)$, $n \to \infty$ being along $\mathbb{N}_{f,x}^{o}$.

5.3 CONVERGENCE IN THE COMPLEX PLANE

Let $K(\theta)$, $\theta \in (o,\pi/2)$, be the subset of $\mathbb{C}^+$ as in Theorem 5 2 1 We denote the upper half of the set by $K^+(\theta)$ and the lower half by $K^-(\theta)$ ($\mathbb{R}^+$ is not included in any of the sets $K^+(\theta)$ and $K^-(\theta)$) Also, let

$$L^+(\theta) = \{ z \mid \operatorname{Re} z > 0 \text{ and } \arg z = \theta \}, \text{ and}$$

$$L^-(\theta) = \{ z \mid \operatorname{Re} z > 0 \text{ and } \arg z = -\theta \}$$

Throughout this chapter $(K^*(\theta), L^*(\theta))$ denotes any of the pairs $(K^+(\theta), L^+(\theta))$ or $(K^-(\theta), L^-(\theta))$.

THEOREM 5 3.1 Let $\theta \in (o,\pi/2)$. If $f \in B^{**}$ is regular in $K^*(\theta)$ and is continuous on $\Lambda^*(\theta) = \mathbb{R}^+ \cup L^*(\theta)$, then

$$\lim_{n \to \infty} M_{n,z}[f(t)] = f(z), \tag{5.10}$$

for any $z \in K^*(\theta) \cup \Lambda^*(\theta)$.

REMARK 5.3.1 The proof of Theorem 5 3 1 uses the order condition (5.2) only on the set $K^*(\theta)$.

First, we shall prove the following lemma.

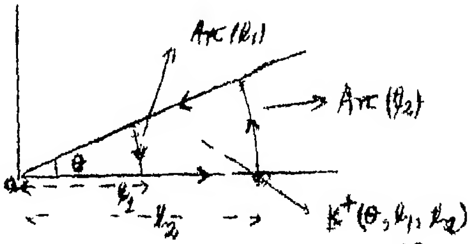
LEMMA 5.3.1 Under the hypothesis of Theorem 5.3.1, for each $z \in L^*(\theta)$, if n is sufficiently large, there hold

$$\lim_{\ell_2 \to \infty} \frac{n^n}{\Gamma(n) z^{n+1}} \int_{Arc(\ell_2)} f(\omega)\, \omega^n e^{-n\omega/z}\, d\omega = 0 \tag{5.11}$$

and

$$\lim_{\ell_1 \to 0} \frac{n^n}{\Gamma(n) z^{n+1}} \int_{Arc(\ell_1)} f(\omega)\, \omega^n e^{-n\omega/z}\, d\omega = 0, \tag{5.12}$$

where $Arc(\ell_1)$ and $Arc(\ell_2)$ are the two arcs of the boundary of the set $K^*(\theta,\ell_1,\ell_2) = \{z \in K^*(\theta) | \ell_1 \leq |z| \leq \ell_2, \ell_2 > \ell_1 > 0\}$



PROOF OF THE LEMMA Let $z = xe^{i\theta}$. For large $|\omega|$, using (5.2), $|f(\omega)| \leq A_1 e^{B|\omega|}$ for some constant A_1. Hence, for sufficiently large ℓ_2,

$$\left| \frac{n^n}{\Gamma(n) z^{n+1}} \int_{Arc(\ell_2)} f(\omega)\, \omega^n e^{-n\omega/z}\, d\omega \right|$$

(5.13)

$$\leq \frac{n^n}{\Gamma(n) x^{n+1}} \int_0^\theta A_1 \ell_2^{n+1} e^{-\ell_2 \{(n/x)\cos(\theta-\alpha) - B\}}\, d\alpha$$

$$\leq \frac{A_1 \theta n^n}{\Gamma(n) x^{n+1}} \ell_2^{n+1} e^{-\ell_2((n/x)\cos\theta - B)}.$$

For n sufficiently large there holds $(n/x)\cos\theta - B > 0$ Hence the right hand side of (5.13) approaches zero as $\ell_2 \to \infty$. This proves (5.11).

Similarly for sufficiently small ℓ_1, using $|f(\omega)| \leq A_2 |\omega|^{-b}$, we have

$$\left| \frac{n^n}{\Gamma(n) z^{n+1}} \int_{Arc(\ell_1)} f(\omega)\, \omega^n e^{-n\omega/z}\, d\omega \right|$$

(5.14)

$$\leq \frac{n^n}{\Gamma(n) x^{n+1}} \int_0^\theta A_2 \ell_1^{(n+1)-b} e^{-n((\ell_1/x)\cos(\theta-\alpha))} d\alpha$$

$$\leq \frac{A_2 \theta}{\Gamma(n) x^{n+1}} \ell_1^{(n+1)-b} e^{-n(\ell_1/x)\cos\theta}.$$

Since for all n sufficiently large, $n + 1 - b > 0$, taking limit as $\ell_1 \to 0$, the right hand side of (5 14) approaches zero, which completes the proof of the lemma.

PROOF OF THE THEOREM We shall only prove the theorem when $K^*(\theta) = K^+(\theta)$, the case $K^*(\theta) = K^-(\theta)$ being similar. Further, to prove the theorem, clearly, it is sufficient to consider the case when $z \in L^+(\theta)$. Let C denote the boundary of $K^+(\theta, \ell_1, \ell_2)$, where ℓ_1 and ℓ_2 are such that $z \in C$. By the hypothesis on f, f is regular in $K^+(\theta, \ell_1, \ell_2)$ and continuous on C. Hence, by the Cauchy theorem,

$$\int_C f(\omega)\,\omega^n e^{-n\omega/z}\, d\omega = 0$$

Hence,

$$\frac{n^n}{\Gamma(n) z^n}\Big\{ \int_{\ell_1}^{\ell_2} f(t) t^n e^{-nt/z} dt + \int_{Arc(\ell_2)} f(\omega)\,\omega^n e^{-n\omega/z}\, d\omega$$

$$(5.15) \qquad - \int_{Arc(\ell_1)} f(\omega)\,\omega^n e^{-n\omega/z}\, d\omega$$

$$- \int_{\ell_1}^{\ell_2} f(te^{i\theta})(te^{i\theta})^n e^{-nte^{i\theta}/z} e^{i\theta}\, dt\Big\} = 0$$

Letting $\ell_2 \to \infty$ and $\ell_1 \to 0$ in (5 15), from Lemma 5.3.1, we get

$$\frac{n^n}{\Gamma(n) z^{n+1}} \int_0^\infty f(t) t^n e^{-nt/z}\, dt$$

$$(5\ 16) \qquad = \frac{n^n}{\Gamma(n) z^{n+1}} \int_0^\infty f(te^{i\theta})(te^{i\theta})^n e^{-nte^{i\theta}/z} e^{i\theta}\, dt$$

PROOF. Let K be a compact subset of $K^+(\theta)$. Then there exist positive ℓ_1, ℓ_2 and δ such that $K \subset K^+(\theta-\delta, \ell_1, \ell_2)$. Let L_1 and L_2, respectively denote the points of $\mathbb{R}^+$ distant ℓ_1 and ℓ_2 from the origin. Also, let Z_1 denote the point $\ell_1 e^{i(\theta-\delta)}$ and Z_2 the point $\ell_2 e^{i(\theta-\delta)}$. Since,

$$\lim_{n \to \infty} M_{n, z_0}[f(t)] = f(z_0),$$

uniformly for all z_0 belonging to the segments L_1L_2 and Z_1Z_2, we can write, for some constant M_1 independent of z_0 and n,

$$(5.18) \qquad |M_{n, z_0}\ [f(t)]\ | \leq M_1, \quad z_0 \in L_1L_2 \cup Z_1Z_2.$$

Now, assume that z_0 lies on the arc L_1Z_1. Then $z_0 = \ell_1 e^{i\alpha}$ where $0 < \alpha < \theta - \delta$. Since $f \in B^{**}$, for some constants A_1, A_2, we have

$$|f(\omega)| \leq A_1 e^{B|\omega|} + A_2 |\omega|^{-b},$$

for all $\omega \in K^+(\theta)$. Therefore, by (5.17),

$$|M_{n, z_0}\ [f(t)]| = \left|\frac{n^n}{\Gamma(n)\, \ell_1^{n+1}} \int_0^\infty f(te^{i\alpha})\, t^n\, e^{-nt/\ell_1}\, dt\right|$$

$$\leq \frac{n^n}{\Gamma(n)\, \ell_1^{n+1}} \int_0^\infty |f(te^{i\alpha})|\ t^n\, e^{-nt/\ell_1}\ dt$$

$$\leq \frac{n^n}{\Gamma(n)\, \ell_1^{n+1}} \int_0^\infty [A_1 e^{Bt} + A_2 t^{-b}]\ t^n e^{-nt/\ell_1}\, dt$$

$$= A_1 M_{n, \ell_1}\ [e^{Bt}] + A_2\, M_{n, \ell_1}\ [t^{-b}], \quad z_0 \in \text{Arc } L_1Z_1$$

Since the right hand side is independent of z_o, it follows from (1.6) and (1.10) that for all $z_o \in \text{Arc } L_1Z_1$ and for some constant M_2 independent of z_o and n,

$$(5.19) \qquad |M_{n,z_o}[f(t)]| \leq M_2$$

In a similar manner it can be shown that when $z_o \in \text{Arc } L_2Z_2$, for some constant M_3 independent of z_o and n,

$$(5.20) \qquad |M_{n,z_o}[f(t)]| \leq M_3$$

Hence, from (5.18) - (5.20), the maximum modulus principle yields

$$(5.21) \qquad |M_{n,z_o}[f(t)]| \leq M,$$

for all z_o $K^+(\theta-\delta,\ell_1,\ell_2)$, M being a constant independent of z_o and n.

The result for $K^+(\theta)$ now follows by Theorem 5.3.1 and Vitali's convergence theorem The proofs for $K^-(\theta)$ and $K(\theta)$ being similar are omitted. This completes the proof.

The following consequence of Theorem 5.3.2 is now clear.

COROLLARY 5.3.2 Let $f \in B^{**}$ be regular in $\mathbb{C}^+$ then uniformly on each compact subset of $\mathbb{C}^+$, there holds

$$\lim_{n \to \infty} M_{n,z_o}[f(t)] = f(z_o).$$

5.4 EFFECT OF SINGULARITIES ON THE REGION OF CONVERGENCE

In this section we consider an isolated singularity ω_o of the function $f(\omega)$ and then determine the corresponding region of convergence of $M_{n,z}[f(t)]$ to $f(z)$ as $n \to \infty$. The following theorem is the main result of the section.

THEOREM 5.4.1 Let $\theta \in (o, \pi/2)$. If $f \in B^{**}$ is regular in $K^*(\theta)$, excepting at an isolated singularity $\omega_o \in K^*(\theta)$, and is continuous on $\Lambda^*(\theta) = \mathbb{R}^+ \cup L^*(\theta)$, then

$$\lim_{n \to \infty} M_{n,z}[f(t)] = f(z)$$

for all $z \in K^*(\theta) \cup \Lambda^*(\theta)$ which satisfy the inequality

$$(5.22) \qquad \left|\frac{\omega_o}{z}\right| \cos(\beta - \alpha) > 1 + \ln\left|\frac{\omega_o}{z}\right|,$$

where $\beta = \arg \omega_o$ and $\alpha = \arg z$.

If $\alpha < \beta$, the proof follows from Theorem 5.3.1. To prove the theorem when $\alpha > \beta$, we require the following.

PROPOSITION 5.4.1. Let C_r denote the circle with centre ω_o and radius r. Then with f as in Theorem 5.4.1 there holds

$$(5.23) \qquad \lim_{n \to \infty} \frac{n^n}{\Gamma(n) z^{n+1}} \int_{C_r} f(\omega)\, \omega^n e^{-n\omega/z}\, d\omega = 0,$$

provided z satisfies the inequality (5.22) and r is sufficiently small.

PROOF OF PROPOSITION 5 4.1 . Let $M = \max_{\omega \in C_r} |f(\omega)|$. Then,

$$\left| \frac{n^n}{\Gamma(n) z^{n+1}} \int_{C_r} f(\omega)\omega^n e^{-n\omega/z} d\omega \right|$$

$$\leq \frac{n^n}{\Gamma(n)|z|^{n+1}} M(|\omega_o| + r)^n e^{-n \min_{\omega \in C_r} \mathrm{Re}(\omega/z)} 2\pi r$$

$$\leq \frac{n^n}{\Gamma(n)|z|^{n+1}} 2\pi r\, M(|\omega_o|+r)^n e^{-n(1/|z|)(|\omega_o|\cos(\beta-\alpha)-r)}$$

$$\cong \frac{\sqrt{2\pi}\, r\, M\, n^{1/2}}{|z|} \left(\frac{|\omega_o|+r}{|z|}\right)^n e^{-n(|\frac{\omega_o}{z}|\cos(\beta-\alpha)-1-\frac{r}{|z|})},$$

(by the Stirling's formula)

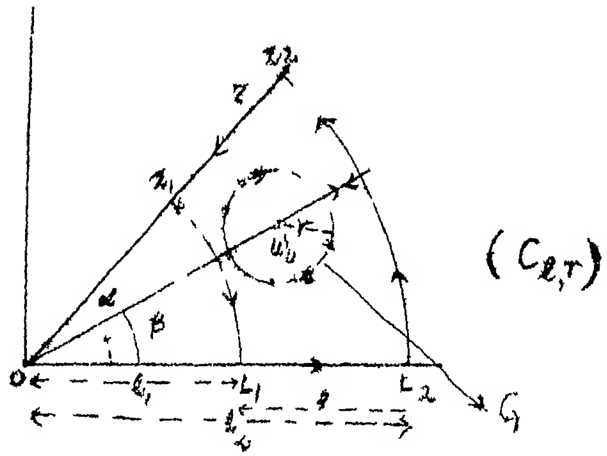
$$= \frac{\sqrt{2\pi}\, r\, M\, n^{1/2}}{|z|} \exp\left(-n\left\{-\ln\frac{|\omega_o|+r}{|z|} - 1 + \left|\frac{\omega_o}{z}\right|\cos(\beta-\alpha) - \frac{r}{|z|}\right\}\right).$$

Now, if r is sufficiently small and

$$\left|\frac{\omega_o}{z}\right| \cos(\beta-\alpha) > 1 + \ln\left|\frac{\omega_o}{z}\right|,$$

then $\left|\frac{\omega_o}{z}\right| \cos(\beta-\alpha) + \left\{-1 - \ln\frac{|\omega_o|+r}{|z|} - \frac{r}{|z|}\right\} > 0$, and therefore last expression converges to zero as $n \to \infty$. This completes the proof.

PROOF OF THEOREM 5.4.1 Let $\alpha > \beta$. Consider the closed contour $C_{\ell,r}$ as shown in the figure below



By Cauchy's theorem,

$$\int_{C_{\ell,r}} f(\omega)\, \omega^n e^{-n\omega/z}\, d\omega = 0.$$

Therefore,

$$\frac{n^n}{\Gamma(n) z^{n+1}} \Big\{ \int_{L_1L_2} f(t) t^n e^{-nt/z}\, dt + \int_{\text{Arc } L_2Z_2} f(\omega)\, \omega^n e^{-n\omega/z}\, d\omega$$

$$- \int_{C_r} f(\omega)\, \omega^n e^{-n\omega/z}\, d\omega + \int_{Z_2Z_1} f(\omega)\, \omega^n e^{-n\omega/z}\, d\omega$$

$$+ \int_{\text{Arc } Z_1L_1} f(\omega)\, \omega^n e^{-n\omega/z}\, d\omega \Big\} = 0.$$

Letting $\ell_1 \to 0$ and $\ell_2 \to \infty$ and making use of Lemma 5.3.1, we obtain

$$M_{n,z}[f(t)] - \frac{n^n}{\Gamma(n) z^{n+1}} \int_{C_r} f(\omega)\omega^n e^{-n\omega/z}\, d\omega = M_{n,|z|}[F(t)],$$

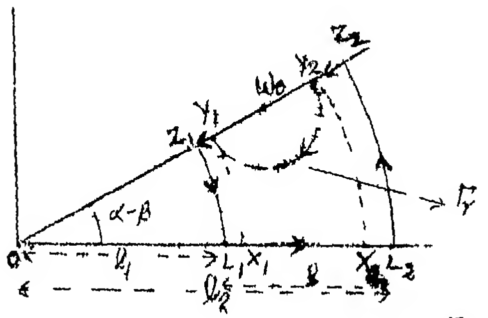
where $F(t) = f(te^{i\alpha})$

From Proposition 5 4.1, the second term on the left hand side of the above relation tends to zero as $n \to \infty$.

From Theorem 1.5.1 we therefore have

$$\lim_{n\to\infty} M_{n,z}\,[f(t)] = \lim_{n\to\infty} M_{n,|z|}\,[F(t)] = F(|z|) = f(z).$$

When $\alpha = \beta$, we consider the contour $C^*_{\ell,r}$ (shown in the figure below) where the radius r of the semicircle Γ_r is taken to be sufficiently small.



$(C^*_{\ell,r})$

Again, by Cauchy's theorem, $\displaystyle\int_{C^*_{\ell,r}} f(\omega)\,\omega^{n}\,e^{-n\omega/z}\,d\omega = 0.$

Therefore,

$$\frac{n^{n}}{\Gamma(n)z^{n+1}}\Big\{\int_{L_1L_2} f(t)t^{n}e^{-nt/z}\,dt + \int_{\text{Arc } L_2Z_2} f(\omega)\omega^{n}\,{}^{\omega/z}\,d\omega$$

$$+ \int_{Z_2Y_2} f(\omega)\,\omega^{n}\,e^{-n\omega/z}\,d\omega + \int_{\Gamma_r} f(\omega)\omega^{n}\,e^{-n\omega/z}\,d\omega$$

$$+ \int_{Y_1Z_1} f(\omega)\,\omega^{n}\,e^{-n\omega/z}\,d\omega + \int_{\text{Arc } Z_1L_1} f(\omega)\,\omega^{n}\,e^{-n\omega/z}d\omega\Big\} = 0$$

Letting $\ell_1 \to 0$ and $\ell_2 \to \infty$ and making use of Lemma 5.3.1 we obtain

$$M_{n,z}\,[f(t)] + \frac{n^{n}}{\Gamma(n)z^{n+1}}\int_{\Gamma_r} f(\omega)\,\omega^{n}\,e^{-n\omega/z}\,d\omega = M_{n,|z|}\,[F(t)],$$

where $\quad F(t) = \begin{cases} f(te^{i\beta}), & t \notin [x_1, x_2] \\ 0 \quad , & t \in [x_1, x_2] \end{cases}$

From the proof of Lemma 5.4.1 it is clear that the second term on the left hand side of the above equality tends to zero as $n \to \infty$ From Theorem 1 5.1 we therefore have

$$\lim_{n \to \infty} M_{n,z}[f(t)] = \lim_{n \to \infty} M_{n,|z|} \; F(t) \quad = F(|z|) = f(z).$$

This completes the proof of the theorem.

The case when $f(\omega)$ has more than one singularity can also be treated in a similar manner.

5.5 BEST POSSIBLE REGION OF CONVERGENCE

The object of this section is to show that the region of convergence determined by (5.22) can not be further enlarged. We prove this in

THEOREM 5.5.1 The region of convergence determined by (5.22) in Theorem 5.4.1 is best possible in the sense that if z ($\neq \omega_0$) does not satisfy (5.22), then $M_{n,z}[f(t)]$ may diverge as $n \to \infty$ for some function $f \in B^{**}$ having ω_0 its only singularity.

PROOF Consider the function $f(\omega) = \dfrac{1}{\omega - \omega_0}$, for which $\omega = \omega_0$ is a pole of order one. The residue at $\omega = \omega_0$, of the function $\omega^n e^{-n\omega/z}/(\omega - \omega_0)$ is $\omega_0^n e^{-n\omega_0/z}$ Therefore, with C_r as in Proposition 5.4.1,

$$\left| \frac{n^n}{\Gamma(n) z^{n+1}} \int_{C_r} \frac{\omega^n e^{-n\omega/z}}{\omega - \omega_0} \, d\omega \right|$$

$$= \left| \frac{n^n}{\Gamma(n) z^{n+1}} 2\pi i \, \omega_0^n \, e^{-n \omega_0 / z} \right|$$

$$= 2\pi \frac{n^n}{\Gamma(n) |z|^{n+1}} | \omega_0 |^n e^{-n \operatorname{Re} \omega_0 / z}$$

$$\cong 2\pi \frac{n^n}{\sqrt{2\pi} \, e^{-n} n^{n-1/2} |z|^{n+1}} | \omega_0 |^n e^{-n \left| \frac{\omega_0}{z} \right| \cos(\beta - \alpha)}$$

$$= \frac{\sqrt{2\pi n}}{|z|} \exp \left(n \left\{ \ln \left| \frac{\omega_0}{z} \right| + 1 - \left| \frac{\omega_0}{z} \right| \cos(\beta-\alpha) \right\} \right)$$

Therefore, if $\ln \left| \frac{\omega_0}{z} \right| + 1 - \left| \frac{\omega_0}{z} \right| \cos(\beta-\alpha) \geq 0$, the last expression goes to ∞ as $n \to \infty$ Notice that in this case $\alpha > \beta$ and therefore from the proof of Theorem 5.4.1 it is clear that $M_{n,z} [f(t)]$ diverges as $n \to \infty$. This completes the proof of Theorem 5.5.1.

5.6 SOME FURTHER RESULTS

In Section 5.3 we evolved an approach to establish the convergence of the Post-Widder operators in the open right half plane, for a certain class of functions. In the present section we adapt another approach to establish the same convergence in which Theorem 5 2.1 plays a prominent role. We start with

LEMMA 5.6.1 Let $f \in B^* \cap C^{\infty}(\mathbb{R}^+)$ and let for a set of positive λ's having a limit point 1 there exist functions $g_\lambda \in B^* \cap C(\mathbb{R}^+)$ such that for some $\theta \in (0, \pi/2)$,

$$|f^{(k)}(x)| \leq \frac{k!\, g_\lambda(x)}{(\lambda x \sin\theta)^k}, \quad \text{for all } k \in \mathbb{N},\ x \in \mathbb{R}^+.$$

Then there exists a positive real number M_x and a positive integer n_0 depending on x such that

$$(5.24) \qquad k^{-1} |M_{n,x}^{(k)}[f(t)]|^{1/k} < M_x, \text{ for all } k \in \mathbb{N} \text{ and } n \geq n_0$$

and

$$(5.25) \qquad \limsup_{k \to \infty} k^{-1} |M_{n,x}^{(k)}[f(t)]|^{1/k} \leq (ex\sin\theta)^{-1},$$

uniformly in $n\ (\geq n_0) \in \mathbb{N}_{f,x}$.

PROOF From the second form of the definition of $M_{n,x}$, formally we have

$$|M_{n,x}^{(k)}[f(t)]| \leq \frac{n^n}{\Gamma(n)} \int_0^\infty |f^{(k)}(tx)|\, t^{n+k} e^{-nt}\, dt$$

$$\leq \frac{n^n}{\Gamma(n)} \int_0^\infty \frac{k!\, g_\lambda(tx)}{(\lambda x t \sin\theta)^k}\, t^{n+k} e^{-nt}\, dt$$

$$= \frac{k!}{(\lambda x \sin\theta)^k} \frac{n^n}{\Gamma(n)} \int_0^\infty g_\lambda(tx) t^n e^{-nt}\, dt.$$

But, since $\lim\limits_{n \to \infty} \frac{n^n}{\Gamma(n)} \int_0^\infty g_\lambda(tx)\, t^n e^{-nt}\, dt$

$$= \lim_{n \to \infty} M_{n,x}[g_\lambda(t)] = g_\lambda(x),$$

there exists a positive integer n_o such that for all $n \geq n_o$

(5.26) $$\left| \frac{n^n}{\Gamma(n)} \int_0^\infty g_\lambda (tx)\, t^n e^{-nt}\, dt \right| \leq A_x^{(\lambda)},$$ where

$A_x^{(\lambda)}$ is a constant which depends on x and λ only. Therefore, it is clear that the interchange of the order of integration and differentiation in the first step of the proof is justified and we have

(5.27) $$|M_{n,x}\ [f(t)]| \leq \frac{k!\ A_x^{(\lambda)}}{(\lambda x \sin\theta)^k},\ n \geq n_o.$$

Therefore,

$$k^{-1} |M_{n,x}^{(k)}\ [f(t)]\ |^{1/k} \leq \frac{k^{-1}}{\lambda x \sin\theta} (k!\ A_x^{(\lambda)})^{1/k}.$$

Using $k! \cong \sqrt{2\pi}\ k^{k+1/2}\ e^{-k}$, and making $\lambda \to 1$ this can be written as

(5.28) $$k^{-1} |M_{n,x}^{(k)}\ [f(t)]\ |^{1/k} \leq B_x,$$ for all $k (\geq \text{some } k_o) \in \mathbb{N}$, $n (\geq n_o) \in \mathbb{N}_{f,x}$ and some constant B_x depending on x only.

Again, since by Theorem 4.2 1, $\lim_{n \to \infty} M_{n,x}^{(k)} [f(t)] = f^{(k)}(x)$, we can write

(5.29) $$k^{-1} |M_{n,x}^{(k)} [f(t)]\ |^{1/k} < C_x,$$ for $k = 1, 2, \quad k_o$ and $n (\geq n_o) \in \mathbb{N}_{f,x}$. From (5.28) and (5.29), for all $k \in \mathbb{N}$ and $n (\geq n_o) \in \mathbb{N}_{f,x}$, we have

$$k^{-1}\ |M_{n,x}^{(k)} [f(t)]\ |^{1/k} < M_x,$$ where $M_x = \max (B_x, C_x)$.

This proves (5.24). Again,

$$\lim_{k\to\infty} \sup k^{-1}|M_{n,x}^{(k)}[f(t)]|^{1/k} \leq \lim_{k\to\infty} \frac{k^{-1}(A_x^{(\lambda)})^{1/k}(k!)^{1/k}}{\lambda x\sin\theta}$$

$$= \lim_{k\to\infty} \frac{(\sqrt{2\pi}\, A_x^{(\lambda)})^{1/k}\, k^{1/2k}}{\lambda e x\sin\theta}$$

$$= \frac{1}{\lambda e x\sin\theta}.$$

Thus,

$$\lim_{k\to\infty} \sup k^{-1}|M_{n,x}^{(k)}[f(t)]| \leq (ex\sin\theta)^{-1},$$

uniformly in $n(\geq n_o) \in \mathbb{N}_{f,x}$.

But, since the left hand side is independent of λ, we can make λ tend to 1 in the above limit relation. This gives

$$\lim_{k\to\infty} \sup k^{-1}|M_{n,x}^{(k)}[f(t)]|^{1/k} \leq (ex\sin\theta)^{-1},$$

uniformly in $n(\geq n_o) \in \mathbb{N}_{f,x}$.

This completes the proof of Lemma 5.6.1.

<u>THEOREM 5.6.1</u>. If f satisfies the hypothesis of Lemma 5.6.1, then f has an extension as an analytic function, regular in

$$K(\theta) = \{ z \mid \operatorname{Re} z > 0, |\arg z| < \theta \}$$

and

$\lim_{n\to\infty} M_{n,z}[f(t)] = f(z)$, uniformly on each compact subset of $K(\theta)$.

PROOF : The proof of the theorem is evident in view of Theorem 5.2.1 and Lemma 5 6.1.

CORROLARY 5.6.1 Let $f \in B^{**}$ be regular in $K(\theta)$ $(o < \theta < \pi/2)$ then

$$\lim_{n \to \infty} M_{n,z} [f(t)] = f(z),$$

uniformly on each compact subset of $K(\theta)$.

PROOF : It follows from the hypothesis on f that for each positive $\lambda < 1$, there exist constants $A, B, b > 0$ such that

$$|f(z)| \leq A(e^{B|z|} + |z|^{-b}),$$

for all $z \in \bigcup_{x \in \mathbb{R}^+} \bar{S}_{x, \lambda x \sin\theta}$, where $\bar{S}_{x, \lambda x \sin\theta}$ denotes the closed sphere with centre x and radius $\lambda x \sin\theta$. It follows that for all $x \in \mathbb{R}^+$ and $z \in \bar{S}_{x, \lambda x \sin\theta}$, there holds

$$|f(z) \leq g_\lambda(x),$$

where $$g_\lambda (x) = \{A\{e^{Bx(1+ \lambda \sin\theta)} + [x(1- \lambda \sin\theta)]^{-b}\}.$$

Clearly, $g_\lambda (x) \in B^* \cap C(\mathbb{R}^+)$, and by Cauchy's inequality

$$|f^{(k)}(x)| \leq \frac{k! \; g_\lambda (x)}{(\lambda x \sin\theta)^k}, k \in \mathbb{N}, \; x \in \mathbb{R}^+, \; o < \lambda < 1$$

Hence the result follows from Theorem 5.6.1.

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